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The value of numerical models in quick response assortment planning

Hajnalka Vaagen[1], Stein W. Wallace[2], Michal Kaut[3]

Abstract

In agile supply chains, dependencies in demand for products (in particular correlations), as well as substitution among products, vary substantially, *and,* due to uncertainty in market acceptance, a substantial share of the portfolio item demands follow bi-modal distributions. Typically, advanced heuristics and major simplifying assumptions on these dependencies are needed to reduce the complexity to an appropriate level for analytical solutions of models. By applying a single-period stochastic model to the multi-item substitutable newsvendor problem, we demonstrate that simplifying assumptions on distributions and dependencies can lead to rather poor solutions, and as a consequence, numerical models – despite their obvious inability to produce general data-independent results – have an important role to play in assortment planning. By using a brand name sportswear assortment problem, we show that even when technology and supply chain flexibility allows for continuous information and production updates, the underlying distributional and dependency assumptions used in the planning models are crucial. We notice, though, that the value of substitution is high and compensates, to some extent for lack of information. We have found that expected profit can drop with as much as 30% when simplifications are applied.

Keywords: Substitution, Correlation, Assortment, QR, Newsvendor, Stochastic Programming, Bimodal distributions, Textile apparel, Attribute based planning.

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1. Introduction and relevant literature

Capturing market trends and satisfying customer demand by supplying quality products in very short time is the dominant challenge in modern manufacturing. Rojas and Frein (2009) show that information sharing across the supply chain members is always better than not sharing, even under the condition when a chain member can judge this information as not reliable for assortment and production planning. Furthermore, sophisticated information technology enables fast and accurate information flow. However, when information available at the time of planning is largely qualitative and based on trend estimates, and lead times are pressed to a minimum, it is crucial to know what kind of information to look for, how to interpret it, and how to apply it in portfolio and production planning. Modelling is about understanding which aspects of a real problem to include in a mathematical formulation and which to leave out. Most of the time, the only way to be sure that a certain aspect of a problem can be left out or simplified – still resulting in a useful model – is to investigate explicitly what happens when that aspect is included or left out. A minimal requirement for doing so is to have a tool allowing the aspect to be included in the model. This paper is set in such a framework. We wish to understand whether or not it is acceptable to simplify dependencies in demands and substitution, and still end up with a useful model. To investigate this we need a tool capable of handling the dependencies, and in our case that is a stochastic program with a careful process for scenario generation to capture the dependencies.

Worried over the variety explosion in contemporary markets and its negative impact on own supply chain activities, many suppliers and retailers turn from offering higher variety to a more efficient assortment strategy. Existing retail and supplier practices can handle corporate complexity. However, this is mostly done 'by art and judgment' and by rather substantial difficulties and inconsistencies when facing large problems. On the theoretical side, there is a substantial amount of work on how to achieve agility by enabling responsiveness and robustness to effectively handle the volatile market place with constantly changing product preferences (see for example Ifandouas and Chapman, 2009). From an operations research perspective, Kök *et al.* (2008) give an extensive review on published work treating the assortment planning problem. Much of this work focuses on analytical formulations that require different heuristics and strong assumptions on the demand patterns and inter-item dependencies to achieve solutions. It is our understanding, both from the literature (see among others, Rajaram 2001, Tang and Yin 2007, Kök *et al.* 2008) as well as own observations, that within assortment planning, dependencies in demand for products (in particular correlations) as well as substitution among products, vary substantially. Rajaram says in 2001 that it is important to model demand as a random variable and to estimate parameters so to reflect the "*significantly more complex*" interaction effects in the assortment. For a practical illustration, consider the integration of textile technology and information-communication technology (ICT) to develop new materials with membranes and catalytic reactors for integrated communication systems. Or nanotechnologies used to develop organic-inorganic hybrids to facilitate 'green' properties. Although the final demand is on end-product level, when new products are launched on the consumer market questions like '*How do 'green' properties, shape memory, integrated ICT, and ephemeral seasonal trend attributes affect each other and the market demand for these and other related products?'* naturally arise. It is rather obvious that, due to the different product attributes, complex dependency structures arise in the assortment. Hence, for example assuming that all correlations and/or substitution parameters are homogeneous across the products does not describe the problem well. Also, as we shall argue in Section 2, assuming unimodal distributions is an extremely strong assumption in agile supply chains. Despite Kök *et al.*'s (2008) recognition of the potentially enormous academic contribution in adding rigor and science to the retailers' developed practices – much like it has been done in for example finance –, discussions on the numerical complexity, tractability, *and* applicability of these formulations, as well as empirical tests of the theoretical predictions, are rather vague. To the best of our knowledge, the only papers providing numerical results on assortment planning with real data are Kök and Fisher (2007), Vaagen and Wallace (2008) and Vaagen *et al*. (2009). The Kök and Fisher model is applied at a supermarket chain. The Vaagen and Wallace and Vaagen et al. models treat assortment risk and product substitution for a sports apparel supplier.

A major goal of this paper is to demonstrate that simplifying assumptions applied by some of the analytical formulations can lead to rather poor solutions, and as a consequence, that numerical models – despite their obvious inability to produce general data-independent results – have an important role to play in assortment planning. The world is simply too complex to be properly analyzed using only analytical models. To demonstrate the value of numerical models in assortment planning, we use a stochastic programming (SP) formulation of the problem given by the frequently cited works of Rajaram and Tang (2001) and Netessine and Rudi (2003): newsvendor-inventory planning with substitution among products with jointly distributed demands. Rajaram and Tang provide one of the very few existing numerical tests to the multi-item problem. As such, these numerical examples are useful for a side-by-side comparison of the behaviour of numerical and analytical approaches. We use recently developed concepts and SP models from Vaagen and Wallace (2008) and Vaagen *et al*. (2009). Both papers allow correlations and substitution to vary across products and can handle demands that are not uni-modally distributed. Vaagen and Wallace (2008) develop a numerically tractable stochastic program for the multi-item newsvendor assortment-risk problem. Their model captures the important profit-risk trade-off encountered in real-life production and retail settings. It is small, convex and well-structured with negligible computation times, indicating its potential usefulness in industrial applications. Vaagen et al. (2009) use the demand modeling process developed by Vaagen and Wallace (2008) to further understand the complex dependency patterns in the assortment. Particularly, they show how substitution and correlations among the individual item demands are connected, and discuss modeling challenges of the complex consumer-directed substitution problem.

The contribution of this paper is the evidence provided on the value of having a numerically tractable stochastic programming formulation of the assortment planning problems with complex dependencies. By applying an SP formulation without solution heuristics and simplifying assumptions common in the operations research literature, we provide insights on how commonly adopted assumptions on dependencies and distributions lead to reduced complexity but also potentially rather bad solutions. Particularly, we show that (i) actual profit can drop with as much as 30% when simplifications on substitution patterns are applied, (ii) underlying distributional assumptions are important even when technology and supply chain flexibility allows for continuous information and production updates, we show (iii) the value of SP in operational variety planning by means of product line trimming, and (iv) the value of SP in attribute based assortment planning. We believe, the findings are appealing from a theoretical perspective, as well as for practitioners and software providers in QR assortment planning.

The rest of the paper is organized as follows. In the remaining of this section we discuss relevant work on substitutable newsvendor problems. In Section 2 we present the multi-step demand-modeling process, and the stochastic program designed to handle the numerical complexity of demand distributions and dependencies. Test cases and detailed discussions on the numerical aspects can be found in Section 3. We conclude in Section 4.

Related literature

For extensive reviews on the assortment planning literature we refer to Kök *et al*. (2008) and Mahajan and van Ryzin (1998). Given that we analyse the substitutable newsvendor problem, we provide a brief description on some central findings about that problem, emphasizing underlying assumptions, solution heuristics and parameter simplifications. Substitution refers to customers' willingness to substitute within a particular product category when facing stockout of their first preference.

For single-period two- and multi-item newsvendor formulations, see among others Parlar and Goyal (1984), Pasternack and Drezner (1991), Gerchak et al. (1996), Bitran and Dasu (1992), Khouja *et al*. (1996), Bassok *et al*. (1999), Rajaram and Tang (2001), and Netessine and Rudi (2003). These use exogenous demand models, where customers choose from a set of products, and in case of stockout they might accept an alternative variant according to given substitution probabilities. If the substitute is also out of stock, the sale is lost. The utility based substitution models assume that the consumer assigns a particular utility to each product, and the variant available with highest utility is chosen; allowing for several substitution attempts. See among others Mahajan and van Ryzin (2001), Gaur and Honhon (2006), and Chong *et al*. (2004).

These analytical formulations have the great advantage of being general in their findings and provide great insights and understanding. However, most authors admit that an explicit solution to the problem is difficult to achieve without different heuristics and simplifications on demand patterns and dependencies; such as normality assumptions or the use of average correlation and substitution values. Bassok *et al*. (1999), in their full downward substitution model, assume independently distributed individual demands realized at the beginning of the planning horizon. Rajaram and Tang (2001) develop conditions under which substitutability between two products enables for reduced variability of the *effective demand* D_i^e, consisting of the original demand D_i for product *i* and the substitution demand *from* other items. A service rate heuristic is further evaluated by a second heuristic to approximate an upper bound on profit. The results obtained by Rajaram and Tang (2001) are analytically confirmed by Netessine and Rudi (2003). The underlying assumption here, as well as in Rajaram and Tang, is that substitution is generated and directed by the consumer. Netessine and Rudi enforce direct sales as first decision. However, the dynamics of the true consumer-directed substitution is not considered, and the assumption of only one substitution attempt is strong.

The Rajaram and Tang formulation clearly implies optimal allocation between direct and substitution sales. This means that although it is the customers' willingness to actually accept a different than first-preference product that underlies the substitution, the substitution process is 'controlled' by the retailer or manufacturer. This is a characteristic of the manufacturer-directed substitution problem. For the properties of manufacturer- and consumer-directed substitution processes we refer to Mahajan and Van Ryzin (2001). This potential gap between what the authors claim to formulate and what they actually do, indicates the substitution problem complexity.

Utility based substitutable assortment planning approaches are better suited to capture the true customer choice behaviour. Mahajan and van Ryzin (2001) achieve nearly optimal solutions by applying a sample path analysis and, as such, allowing for a general stochastic process of arrivals. Despite this, these models' limitations make them less useful in agile environments with complex dependencies. The IIA property of MNL models (Independence of Irrelevant Alternatives) requires that the ratio of choice probabilities is independent of the choice set. Further, solutions derived by these formulations clearly define the assortment structure to a particular combination of popular/unpopular products. In other words, the products are defined ex-ante planning to be either popular or unpopular (see also van Ryzin and Mahajan, 1999). Gaur and Honhon (2006) recognize that an assumption of exact knowledge of customer choice makes the substitution problem less complex. More precisely, the authors say "*Another limitation of the locational choice model is with respect to the nature of randomness in customer choice … given the most preferred good of a consumer, the sequence of product selections that the consumer makes is known precisely. It would be natural to generalize the model to allow randomness both in the locations of the most preferred goods of consumers and in customers' sequence of product selection*". However, we could not find any concrete suggestion on how to relax this strong and potentially unrealistic assumption of ex-ante knowledge of popularity; except in Vaagen and Wallace (2008).
The consumer choice model of Chong *et al*. (2004) captures the effect of product substitution in terms of expected demand and demand variability, when making product trimming decisions. Although trimming affects the individual demands as well as the aggregated portfolio demand, dependencies among the individual items are not directly considered.

Probability distributions in consumer choice models (such as Poisson customer arrivals), as well as in the exogenous demand models, are frequently simplified by normal approximations. Mahajan and van Ryzin (2001) point to this being a common practice when applying inventory models in practice. To achieve a solution, the numerical example of Rajaram and Tang replaces dependency descriptors (such as correlations) with average values across the group. Accordingly, we do not know whether the findings are also valid for heterogeneous dependency and demand patterns. Vaagen and Wallace indicate in 2008 that by mis-specifying distributions and correlations, fashion and sports apparel suppliers might introduce internal uncertainty into their planning processes, and hence increase the risk of their future payoffs.

It is also unclear how the parameters required for these models can be appropriately estimated. Despite the large amount of work on assortment optimisation, less attention is given to substitution behaviour estimation (see also Kök *et al*, 2008, for references). Kök and Fisher (2007) present a substitution estimation approach that works when sales summary data is available. Before that, Anupindi *et al*. (1998) develop an estimation model that requires inventory-transaction data. In QR supply chains, assortment decisions are mainly taken in light of uncertainty, when the information available is limited to subjective understanding and aggregated estimates. As such, these estimation methods are less appropriate in QR settings. Vaagen *et al*. (2009) develop a decision independent approach with substitution shares describing the similarity between the items with regard to the demand driver attributes; a method which we believe is better suited for dynamically changing trend driven industries.

2. Model formulation and parameter estimation

Model formulation

Based on observing increasingly complex model formulations and the recognition that multidimensional newsboy models with complex dependencies are analytically difficult to deal with, we analyse whether the numerically approachable SP formulation provides better solutions. We intend to avoid solution heuristics and simplifying assumptions on distributions and substitution patterns, as these in our understanding drive the assortment structure. As such, it is obvious that we are dependent on data; data that cannot be obtained without internal understanding of the specific environments. However, we do not consider this to be a disadvantage, but rather recognition and acceptance of reality.

We solve the problem given by Rajaram and Tang, 2001 (henceforth, RT). As discussed in the literature part, the outcome of the RT model is an optimal allocation between direct and substitution sales; hence we choose to analyse the same problem. The optimal solution implies that the focal part (manufacturer or retailer) controls these values; in other words, it tells the customers how much of the first and lower preferences they may buy. Although this is a simplified case of the true substitution process observed in retail settings, agile suppliers can indeed, to some extent, control the choice process. An example is the textile apparel franchisee. Recall that our focus is on demonstrating the impact of simplifications on, in our understanding, central terms. As such, we do not discuss the modeling aspects and potential solutions of the complex consumer-directed substitution problem; for this, we refer to Vaagen *et al*. (2009).

In the numerically tractable two-stage stochastic program given by Vaagen *et al*., (2009), the first stage consists of the production decisions before observing demand. The second stage (after demand has been realized) optimally allocates direct and substitution sales. See Kall & Wallace (1994) for further information about two-stage stochastic programs. More precisely, the following process is modeled: In a first step, most appropriately in the design phase, the manufacturer/retailer defines the similarity/dissimilarity between the products with regard to the demand driver attributes (may be subjective), and uses this information to establish the substitution measures and correlation matrices. As such, the substitution measure is independent of inventory levels and sales transaction data. Secondly, given substitutability and the established individual item demand distributions, the manufacturer decides the optimal assortment to offer: products to include in the portfolio and their inventory levels. Finally, when the actual customer demand becomes known, the manufacturer assigns first and substitute preferences so as to maximize expected assortment profit, given the initial inventory levels and substitutability matrix. The outcome of this process is the factual substitution.

The model

Sets:

S – set of demand scenarios;

I – set of items in the reference group portfolio

If we do not state otherwise, we use indices with the following meaning $i, j \in I, \; s \in S$

Variables

x_i = production of item i

y^s_i = sale for item i scenario s

z^s_{ij}= substitution sale of item i, satisfying excess demand of item j in scenario s

zt^s_i = substitution sale of item i, satisfying excess demand from all j's in scenario s

w^s_i = salvage quantity for item i in scenario s

Parameters

d^s_i = demand for item i in scenario s

p^s = probability of scenario s

v_i = selling price for item i

c_i = purchasing cost for item i

g_i = salvage value for item i

$\alpha_{ij} \in [0,1]$ = substitutability probability; the probability that the consumer is willing to accept item *j* when actually wanting item *i*

$$Maximize\ Expected\ profit = \sum_{s \in S} p^s \sum_{i \in I} \left(-c_i x_i + v_i y_i^s + v_i\, zt_i^s + g_i w_i^s \right) \quad (1)$$

Subject to:

$$y_i^s + \sum_{j \in I, j \neq i} z_{ji}^s \leq d_i^s \quad \forall i \in I, s \in S \quad (2)$$

$$z_{ij}^s \leq \alpha_{ij} \left(d_j^s - y_j^s \right) \quad \forall i, j \in I, i \neq j, s \in S \quad (3)$$

$$zt_i^s = \sum_{j \in I, j \neq i} z_{ij}^s \quad \forall i \in I,\ s \in S \quad (4)$$

$$w_i^s = x_i - (y_i^s + zt_i^s) \quad \forall i \in I, s \in S \quad (5)$$

$$x_i \geq 0 \quad \forall i \in I \quad (6)$$

$$y_i^s \geq 0 \quad \forall i \in I, s \in S \quad (7)$$

$$z_{ij}^s \geq 0 \quad \forall i, j \in I, i \neq j, s \in S \quad (8)$$

$$zt_i^s \geq 0 \quad \forall i \in I, s \in S \quad (9)$$

$$w_i^s \geq 0 \quad \forall i \in I, s \in S \quad (10)$$

Expected assortment profit from ordinary sales, substitution sales and salvage, over all items and all scenarios, is maximized using Equation (1). Equations (2) state that that total sales for item *i* – coming from primary demand for *i* plus all *j* sales generated by unmet demand for *i* – are constrained by the total demand for item *i.* This constraint can be re-organized as

$$\sum_{j \in I, j \neq i} z_{ji}^s \leq d_i^s - y_i^s \quad \forall i \in I, \forall s \in S$$

stating that substitution sales from item *i* cannot exceed available unsatisfied demand for *i*. Equation (3), for a given *i,* is the upper bound on substitution sale of item *i* for item *j*; that is, excess demand for item *j* with given substitutability probability α_{ij}. For a given *i*, (4) gives the overall substitution sale *i* from all *j*'s. Equation (5) defines the salvage quantity; the quantity of item *i* left after satisfying primary demand and substitution demand from all *j*. Constraints (6), (7), (8) and (9) are non-negativity constraints for the respective variables. Expressions (5) and (10) together imply that the substitution sale of item *i* is limited to the remaining supply of the item; that is,

$$zt_i^s \leq x_i - y_i^s \quad \forall i \in I, s \in S.$$

Parameter estimation

Parameter estimation, and particularly the multi-step demand modelling process, is an essential part of this approach and the SP framework is chosen for its ability to handle the established demand distributions. Furthermore, simultaneous discussion on parameter estimation and assortment optimisation enforces focus on achieving sufficient consistency with regard to information used throughout the estimation and optimisation process.

Below, we treat parameters defining the effective demand for a product *i*, consisting of the original demand (described by mean demand, its variance, and the correlations between demand for item *i* and all other items *j*) *and* the substitution demand from *j* items (driven by substitutability measures; reflecting the similarity between the items with regard to the demand driver attributes). Parameter estimation is based on the works of Vaagen and Wallace (2008) and Vaagen *et al*. (2009). The scenario-based two-step process, proposed by Vaagen and Wallace (2008), models product demand with complex correlation structures. The demand modelling concepts developed by the authors are then used by Vaagen *et al*. (2009) to establish substitution measures in a multi-item newsvendor assortment problem.

In the context of a sports apparel producer, Vaagen and Wallace (2008) handled the 'chaotic' demand patterns (according to Christopher *et al*., 2004) – frequently simplified by both theory and practice – as numerically tractable bi-modal distributions, easily updatable with increased information about which items are more or less accepted by the market. The authors' demand model is based on two major observations. *Firstly*, several previous studies indicate that nearly half of the assortment becomes obsolete, and that only few items stand for a large share of the assortment profit (see among others Raman *et al*., 2001; Vaagen and Halskau, 2005; Vaagen and Wallace, 2008). In other words, there is 'competition' among the products in the portfolio. *Secondly*, in agile supply chains (such as fashion apparel), at the time of assortment decisions, there is limited information about which products become accepted by the market. Hence for many of them the world takes two possible states: *State 1* when accepted, and *State 2* when there is limited market acceptance (Vaagen and Wallace, 2008).

Further, Vaagen and Wallace (2008) point out that the much less uncertain aggregated assortment demand refers to demand across all the 'accepted' items. In other words, aggregated estimates do not consider the potential states of the world but show the potential demand *given* that the *supplier/retailer* offers the right products. Hence, simple distributions are appropriate to describe the state conditioned individual item demand distributions. However, decision makers are frequently exposed to the uncertainty of market acceptance and, hence, the existence of two states of the world, and possibly bi-modal distributions.

The two-step direct-demand modelling process is summarized as follows (for visualization, a two-item illustration is given in figure 1*):*

(1) For the two possible states of the world, the authors do not approximate the complex bi-modal overall distributions, but define the state-conditioned distributions by using aggregated demand data across whatever will become accepted/not accepted by the market. For the items P1 and P1 in the figure, the conditional marginal distributions are $d_1/S1$, $d_1/S2$ and $d_2/S1$, $d_2/S2$ respectively. Further, the authors generate scenarios for each state of the world independently, using some appropriate scenario-generation tool. A

special version of the moment-matching algorithm from Høyland et al. (2003), the method of Kaut and Lium (2007), is used here. This method allows generating scenarios from distributions specified by their marginal distributions and a correlation matrix. The correlations are established by using understanding on the demand drivers between the products ($c_{1,2} \in [-1,+1]$ in the figure).

(2) The overall distribution is then built by connecting all the scenarios by state probabilities that sum up to one (*prob(S1)* and *prob(S2)* in the figure). Under limited information, the two states occur with equal probabilities. This way, the uncertainty in the individual items' market acceptance is captured.

Substitutability between the products is established in a similar manner as correlations, based on understanding how the product attributes drive the similarity between products with regard to customer preferences; accordingly, we also indicate this measure $\alpha_{1,2} \in [0,1]$ in the figure.

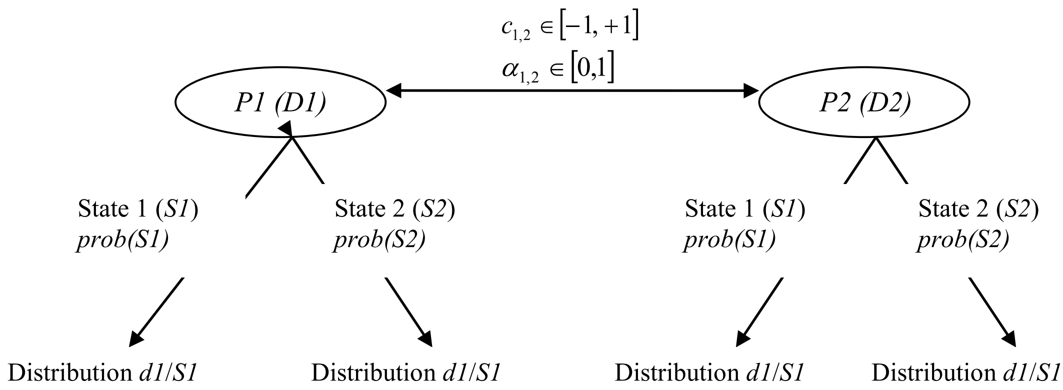

Figure 1 The demand modelling process, illustrated for two items

A major distinction of the works of Vaagen and Wallace (2008) and Vaagen *et al*. (2009) from other works in the literature is the authors' way to approach inter-item dependencies – described by correlation and substitution measures –, by using understanding of the demand driver attributes (might be subjective/qualitative or quantitative), rather than inventory and sales transaction data. Below, we summarize the rules that drive inter-item dependencies.

Product attributes (such as colour) that cause the existence of two states of the world for some items, paired with uncertainty about market acceptance, drive the negative correlations. Similarities on some demand driver attributes, like the existence of specific technical features across products and brand preference, drive the positive correlations. Finally, understanding of the (dis)similarity between the items with regard to demand driver attributes is used to establish the decision independent substitution measures, *a priori substitutability* $\alpha_{ij} \in [0,1]$. This measure indicates the portion of customers willing to replace item *j* with item *i*. The *n* items offered are potential substitutes for each other, with heterogeneous substitutability values.

Vaagen *et al*. (2009) emphasize the distinction between *a priori substitutability* and the true substitution, called *factual substitution*; this latter being a decision dependent outcome of an optimisation process, constrained by unsatisfied demand and the variants available at the moment of customer choice. Further, correlations and substitution measures are connected by a common information base. This is also reflected in the way these measures are established. For illustration, consider the following example. The supplier is to make assortment decisions on two identical apparel models in colours black and navy. Assume the information available for decision making to be: (a) Colour is a strong trend driver, and only black or navy will become popular; this can be described by a strong negative correlation between their demands (for example -0.5); (b) Due to the similarity with regard to 'model', if one becomes popular and faces stock-out, it can partially be substituted by the available one (say with substitutability of 0.2). Given the 'competition' between the products, and given that 'colour' is a much stronger demand driver it is unrealistic to assume high substitutability. Observe how this subjective market understanding is built into both the correlation and substitution measures.

Despite the inability of Vaagen *et al.* (2009) to fully describe dependencies among the substitute choice possibilities by the suggested substitutability matrix, the presented substitution approach, *together* with the modelling process, allows handling the most important dependencies; such as negatively correlated substitute choice possibilities and positively/negatively correlated first and second choice possibilities.

Cost and selling prices are assumed to be homogeneous, and not of strategic importance within this framework. By this we do not state these parameters are not important demand drivers. What we say is that in trend driven supply chains (such as fashion and sports apparel), and within a narrow assortment where competition among the products naturally arises, it is increasingly more difficult to influence customers by price.

Finally, for side-by-side comparison of our results with those obtained by RT, we also define parameter values as given by RT; particularly, we apply similar simplifications on the correlation and substitution values, and on the nature of demand distributions. Some important changes from RT are necessary, though. *Firstly*, with the demand parameters given by the authors the assumption of normal distribution implies the occurrence of negative demands with some probabilities. To avoid this, we do not assume normality but use log-normal distributions with the same means, variances, and correlation values. *Secondly*, RT replace all correlations with an average value, and vary this average between *-1* and *+1,* a simplification leading to impossible correlation matrices (i.e. matrices that are not positive semi-definite) for some of their choices. A necessary condition for a positive semi-definite correlation matrix is $c \geq -\frac{1}{n-1}$, where c is the chosen average correlation and n is the number of items[4]. In a 7-item case, this gives the

[4] A matrix C is positive semi-definite if $x^T Cx \geq 0$ for all vectors x. Taking a vector x with $x_i = 1$ and using the fact that the sum of the elements of the matrix C is equal to the sum of its diagonal plus $n(n-1)$ times the average correlation c, we get

$x^T Cx = \sum_{i,j=1}^{n} c_{ij} = n + n(n-1)c \geq 0$. Re-arranging the inequality then gives the bound $c \geq -\frac{1}{n-1}$.

lower-bound of *-0.166* on the average correlation. We take this constraint in consideration under our parametric analysis.

3. Test cases to demonstrate the value of stochastic programming in assortment planning

In this section we demonstrate the value of stochastic programming in assortment planning. Particularly, we show that simplifying assumptions on dependencies and distributions lead to assortment solutions and profits that substantially differ from those obtainable when no substantial simplifications are made. Many existing decision support tools and software require major simplifications on these terms. We believe there is a need to numerically highlight these shortcomings and use model formulations that actually handle this type of complexity; especially important in QR supply chains, such as fashion and sports apparel. We *first* apply a case from sports apparel, where real company data is used to directly define and estimate model parameters. *Secondly*, for side-by-side comparison of the behaviour of analytical and numerical model formulations, we also solve the retail merchandizing numerical examples provided by Rajaram and Tang (2001).
Our model has been implemented in AMPL with CPLEX as the underlying solver; see http://www.ilog.com/ for details on both systems. Since this is a linear programming model of moderate size, the solution times are negligible.

The sports apparel assortment problem with 15 items

We implement an assortment problem with 15 items from a leading brand name sportswear supplier. The variants within the group are distinguished by the demand driver attributes style and colour, while the specific technical attributes are assumed to be necessary to release sales. Substitution is considered within the group. This case is also used in Vaagen and Wallace (2008)[5], in a related but different setting, newsvendor-risk modelling with no substitution.

The demand and dependency measures (correlations and substitutability) are established as given in Section 2. Company data and understanding show that there are complex and heterogeneous dependencies among the individual items *and* that about half of the

[5] Vaagen and Wallace (2008) use the same 15-item assortment case to say something about the profit-risk trade-off encountered under different assortment decisions. Although this work identifies distributional and correlation model error as a significant risk driver, the numerical analyses differ on several levels. *Firstly*, the Vaagen and Wallace paper focuses on risk modelling. *Secondly*, based on Vaagen and Wallace, exclusively, there is no way to conclude on the value of SP in assortment planning. Particularly, there is no way to conclude whether identical results could have been obtained by analytical formulations. This is simply because, to the best of our knowledge, the Vaagen and Wallace formulation is the first in the literature on the multi-item assortment risk problem, and it is an SP formulation. *Finally*, substitution is not considered by Vaagen and Wallace. As a consequence, this work is not suited to say something useful on the value of substitution under distributional/correlation model error.

portfolio item demands are bi-modally distributed. Furthermore, based on empirical evidence, the state conditioned demands are assumed to be log-normally distributed. We assume this case describes the 'true' problem, and is denoted *Bimodal*. To understand the effects of mis-specifying the uncertainty and dependencies, we define additional situations, incorrectly assuming uni-modality (rather than bi-modality) in demand distributions. Precisely, log-normality is used, with means and variances as in the bimodal distributions. We analyse this under three different dependency patterns: a correlation matrix with all entries equal to zero, a correlation matrix with all entries equal to 0.5, and the assumed 'true' matrix with heterogeneous values. The incorrect uni-modal cases are denoted *LogN-c=0*; *LogN-c=0.5* and *LogN-c=true.* The necessary condition for a positive semi-definite correlation matrix indicates that homogeneous negative correlation values are not to be below -*0.071*, that is, nearly zero; hence, our choice to study the zero-correlation case. The four test cases are given in table 1.

Profit and production levels are analyzed while varying a homogeneous substitutability in [0, 0.5], in addition to using our estimated 'true' matrix (denoted *mix*). Average substitutability values over 0.5 are unrealistic across a group of 15 items. Note that the test-cases with the true correlation matrix (*Bimodal* and *LogN-c=true*), having both negative and positive values, are not well suited to directly conclude on the effect of correlations. To do this, we compare the test results of *LogN-c=0 and LogN-c=0.5*.

Table 1 Test cases

Subcase	*Marginal distributions*	*Correlation values - c*
1 Bimodal	bimodal	True matrix
2 LogN-c=0	uni-modal	assumed zero correlations
3 LogN-c=0.5	uni-modal	homogeneous values *c = 0.5*
4 LogN-c=true	uni-modal	True matrix

Test results

Although our test results confirm previous qualitative findings (Rajaram and Tang 2001, Netessine and Rudi 2003), they say nothing about the effect of mis-specifying distributions, correlations and substitution measures. For this, decisions based on incorrect assumptions must be measured relative to the true distributions and dependencies. We demonstrate this below, by defining two types of mis-specifications: "Substitution model error" and "Distributional model error", the latter covering both correlations and modality of distributions. Finally, we demonstrate the numerical formulation's value in operational variety planning.

Substitution model error

Here we analyse the effects of mis-specifying substitutability, replacing the 'true' substitution willingness (*mix* matrix) by its average (*0.15*). For all test-cases, the expected profit under the average substitutability (as measured within the model) is found to be higher than the expected profit using the mix matrix; see figure 2. The true *Bimodal* case results in almost 15% higher expectation under the average substitutability than under the mix matrix (***2 478 901*** versus ***2 166 684***). However, this does not provide the true picture. If production decisions correspond to the average substitutability, but the true substitution willingness describes the world, decision makers will end up over 30% below their expectation (***1 705 694*** versus *2 478 901)*. Low production levels under average substitutability (*2498* versus *3112* units; figure 3) − implying low flexibility to adapt to changes when the world turns out to be the *mix* substitution matrix − explain the large error in expectations.

The substantially lower error (12 %) when comparing *LogN-c=0.5* within *Bimodal* is due to the strong positive correlation among the items. Substitutability cannot truly be leveraged on, as the products mostly face stockout or overproduction simultaneously. The optimal plans suggest almost equally high production levels (*2995* versus *3087*) and, hence, we observe reduced profit loss. *The effects of substitution, and that of mis-specifying it, are less significant when the products are strongly positively correlated.*

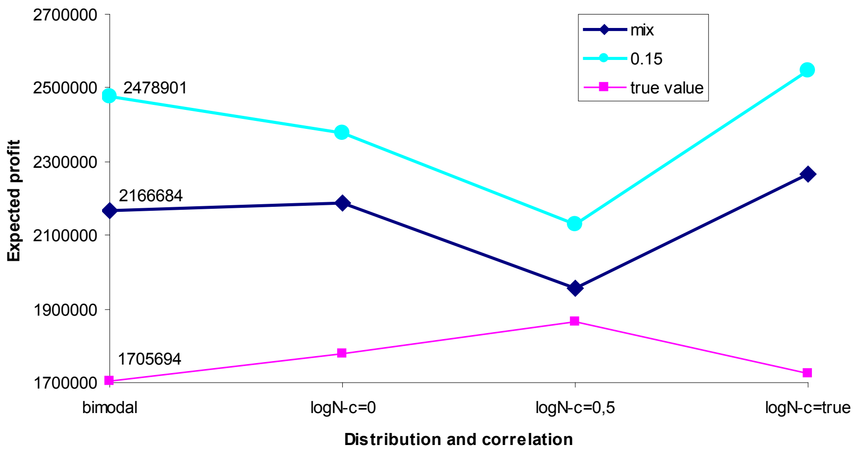

Figure 2 Expected profit under the 'mix' substitutability matrix and its average 0.15, evaluated for the four subcases. The lower curve shows the true profit when the average substitutability 0.15 is assumed, but the world is described by the Bimodal subcase and the mix matrix.

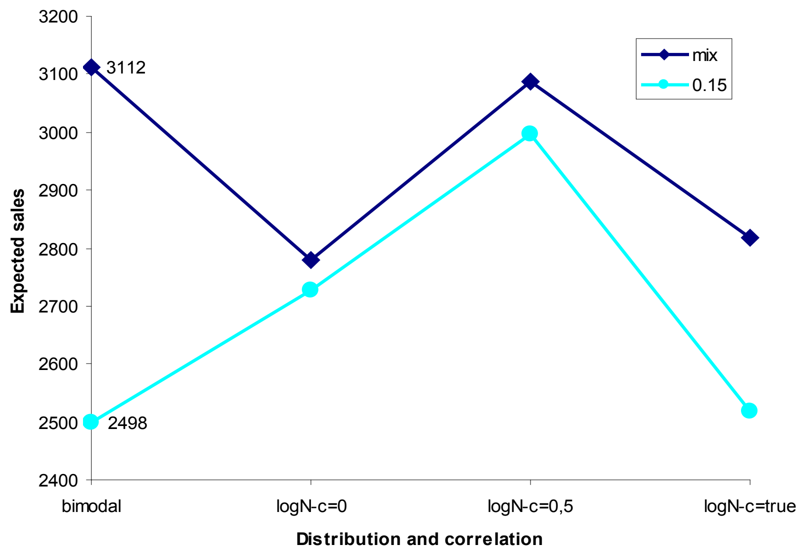

Figure 3 Order quantity for the for subcases, under the assumptions of mix matrix and the average 0.15

Distributional model error

Figure 4 illustrates the effects of mis-specifying distributions by incorrectly assuming uni-modality when the world is described by bimodal distributions. We evaluate production decisions obtained from the uni-modal cases *LogN-c=0*; *LogN-c=0.5* and *LogN-c=true* within *Bimodal*. Profit loss, then, is evaluated by comparing the results with the optimal solution of *Bimodal*, for corresponding substitution values.

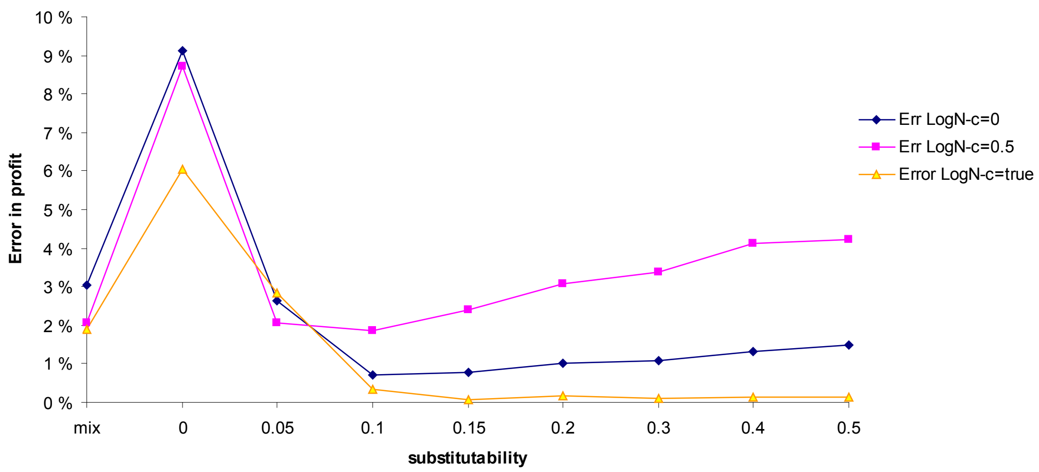

Figure 4 Minimal profit error versus substitution for the uni-modal subcases measured within the true Bimodal subcase

We see, again, that substitution partially compensates for lack of information. Further, under no substitution and for most substitution values, the error is lowest when at least the true correlations are described; that is, *LogN-c=true* is used. The heterogeneous correlation values in our 'true' matrix, and especially the negative correlations, imply some hedging, contributing to reduction in error when *Bimodal* actually describes the world.

The evolution of distributional model errors while updating information and re-optimising production levels accordingly

Reaction to real-time customer orders is a major focus area of QR planning. Here we demonstrate that even when technology and supply chain flexibility allows for continuous information and production updates, the underlying distributional and dependency assumptions used in the planning models are crucial.

The performance error in the section above is studied under maximal uncertainty; assuming that the two states of the world describing the demand uncertainty (Section 2) occur with equal probabilities. In a next step, we measure the negative effects of incorrectly assuming uni-modality as information about the items' market acceptance is revealed. In other words, loss in expected profit is evaluated while increasing the belief about *State 1* (*probability (State 1) > 0.5*), for the uni-modal cases *LogN-c=true* and *LogN-c=0*. Production decisions are re-optimized for all cases at hand, and for all information levels investigated. The results are summarized by figures 5 and 6. Figure 6 gives the profit loss when, in addition to the incorrect uni-modal assumption, correlation values are also incorrect (assumed to be zero). Although substitution partially eliminates the distributional model error, figure 5 shows that, even with very accurate information (*probability (State1) =0.9*), *and* even when the true correlation matrix is used, distributional assumptions are important; the error is up to 16%.

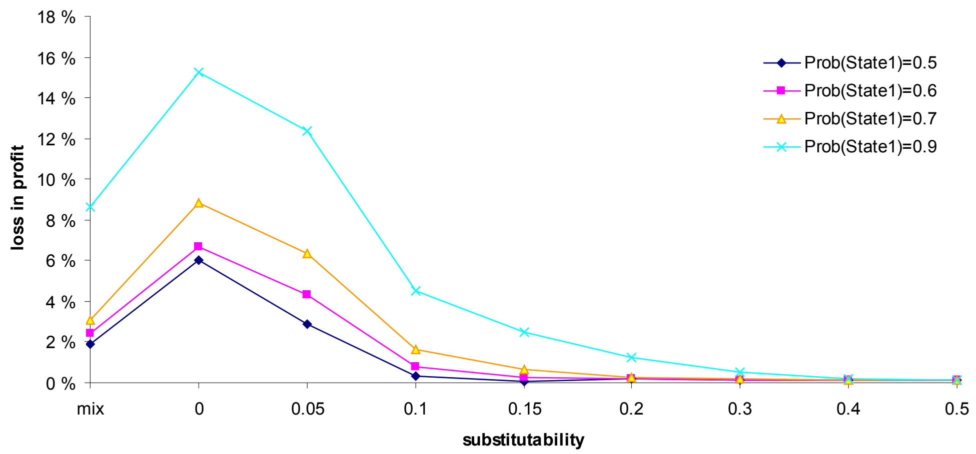

Figure 5 Percentage loss in expected profit under subcase LogN-c=true versus *Bimodal; – evaluated for increasing belief in State 1 for different substitution values.*

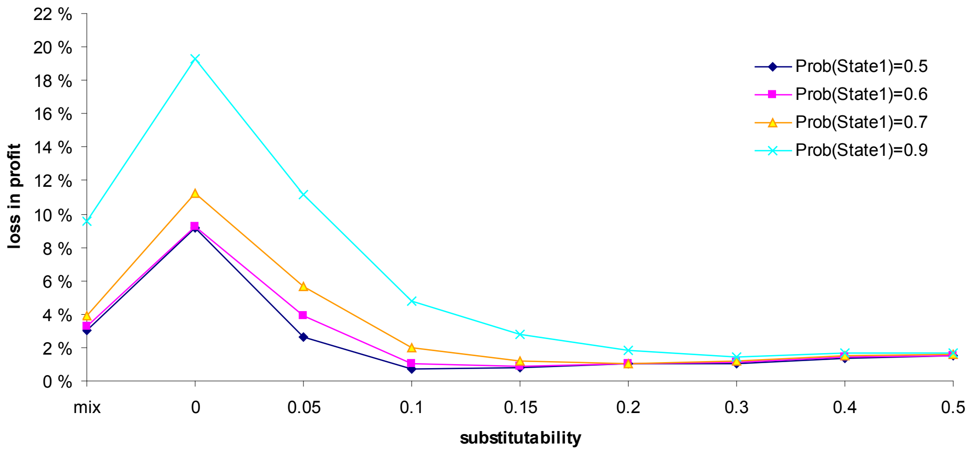

Figure 6 Percentage loss in expected profit under subcase LogN-c=0 versus *Bimodal; – evaluated for increasing belief in State 1 for different substitution values.*

The substantial rise in error under almost full information is caused by the nature of the bimodal distributions. Using the same example, without substitution though, Vaagen and Wallace showed in 2008 that information needs to be very accurate (over 90% probability for one of the states) to lead to important assortment changes, by dropping hedging. We now confirm this, under a different but related problem. Drop in production levels under almost perfect information for *Bimodal* reduces costs substantially; hence, increases expected profit. This, in order, leads to substantial increase in error when uni-modal distributions are incorrectly assumed.

The substitutable portfolio structure

In this section we illustrate the numerical formulation's usefulness in operational variety planning and product line trimming. Table 2 gives the individual item production quantities for the bimodal subcase, when varying substitutability. For better visualization, production levels under 20 units are eliminated. We observe that the optimal portfolio profit implies trimming some of the products. These products, individually, contribute to the performance and are initially included in the portfolio. However, their substitutability to other products makes them redundant.

Finally, observe the substantial difference in assortment decisions under true substitutability (*mix*) and when assuming the average *0.15*; indicating that the problems solved are indeed different.

Table 2 Changes in production quantities for different substitutability values

	Production quantity per item															
substitutability	***1***	***2***	***3***	***4***	***5***	***6***	***7***	***8***	***9***	***10***	***11***	***12***	***13***	***14***	***15***	***total***
mix	93	579	486	56	197	84	197	243	194	-	172	784	-	-	21	3105
0	389	393	393	152	152	152	150	153	153	150	389	347	334	334	358	3999
0.05	354	384	378	161	160	162	167	163	163	52	402	181	45	43	59	2874
0.1	220	410	338	102	111	120	299	190	190	-	179	311	-	-	161	2632
0.15	327	426	424	-	-	-	317	115	123	-	274	376	-	-	115	2498
0.2	280	512	508	-	-	-	333	-	-	-	314	468	-	-	-	2415
0.25	155	627	619	-	-	-	249	26	-	-	207	561	-	-	-	2444
0.3	79	732	734	-	-	-	128	-	-	-	100	638	-	-	-	2411
0.35	-	833	832	-	-	-	39	-	-	-	32	658	-	-	-	2393
0.4	-	839	852	-	-	-	-	-	-	-	-	681	-	-	-	2372
0.45	-	810	875	-	-	-	-	-	-	-	-	699	-	-	-	2383
0.5	-	801	831	-	-	-	-	-	21	-	-	752	-	-	-	2404

We conclude on our test-case, by stating that the assortment is indeed sensitive to the nature of demand distributions/correlations and substitution measures. The value of substitution is high and compensates, to some extent, for distributional model error; however, only when the 'true' substitution pattern is captured. We have found up to 30% drop in expected profit when simplifications are applied. Further, we show that the underlying distributional assumptions are important even when technology and supply chain flexibility allows for continuous information and production updates.

Finally, we point out that although model parameters are established by using real data and industry understanding, the estimation and optimisation approaches are not yet validated by further empirical studies. However, direct comparison of our results and those obtained by the present company practices (an ex-post season evaluation) reveal a profit increase above 30% by applying the approach presented here (precisely, a net profit of *1 387 033* versus our result of *2 166 684*). Planning at the case company is based on deterministic ERP (Enterprise Requirement Planning), with continuous updates in light of real-time customer orders and customer behaviour evaluation. Although the decision makers' subjective understanding reveals substantial intuitive knowledge on the uncertainty and dependencies, this is not directly incorporated into the planning process, but it is done by art-and-judgment. This is due to the simple fact that there is no commonly available software to treat this complexity.

The two- and 7-item examples from Rajaram and Tang (2001)

Although applying the SP formulation to the RT numerical example does not provide empirical evidence on how complex distributions and dependencies affect the performance measures and factual substitution, and nothing on what would have happened if the true correlation and substitution values were used (as we only have simplified data to work with), we find it useful to provide a side-by-side comparison of the numerical and analytical models' behaviour.

For the data applied in the analysis we refer to Rajaram and Tang (2001). Inventory and profit levels are investigated for varying substitutability in [0; 1] and varying demand correlations in [-1; 1], for different levels of demand uncertainty (Low, Medium, High and Mixed variation). Precisely, two independent sets of analyses are performed: one with fixed substitutability and a second with fixed correlation value. Although the

concrete results for the 7-item example are not stated by RT, we present them for clarity of our conclusions. Here we perform one set of analyses with substitutability fixed at *0.4* and a second with correlation fixed at *-0.1*. Recall that we follow the necessary condition for positive semi-definite correlation matrices and do not apply values below *–0.166.*
To avoid negative demands, we do not follow the normality assumption of RT but use log-normal distributions. The low-variation data allows for normality though (as the probability of negative demand is extremely low). Hence for this case we perform the analysis under normal as well as log-normal assumptions (denoted LowN and Low).

Test results

The proposed numerical formulation provides explicit solutions to the examples of RT, and our results confirm, again, the qualitative findings of Rajaram and Tang (2001) and Netessine and Rudi (2003). Under substitution, the profit is decreasing in any correlation value when order quantities are adjusted optimally as the correlations change. This decrease is largest under high variation, and high positive correlation between the items. High positive correlation implies that the individual items follow similar demand patterns. They become popular, unpopular, or stable at the same time. There is either overproduction of all items, or underproduction combined with substantial unsatisfied demand. Substitution cannot be leveraged on, despite substitutability. Substitution is more beneficial under negatively correlated demands. An illustrative summary of these observations in the 7-item setting can be seen in figure 7. The potential *profit rise* by increasing substitutability from *zero* to *0.2* is given for two distinct correlation values, namely *0.4* and *-0.1*. The profit rise for a given correlation can be interpreted as the potential value of strategically designing the products to be each others' partial substitutes with value *0.2*. Observe that this value is substantially better under negatively correlated demands, especially when there is also *High* uncertainty in demand. This can also be seen in figure 8, showing expected profit levels versus substitutability for correlation *-0.1*.

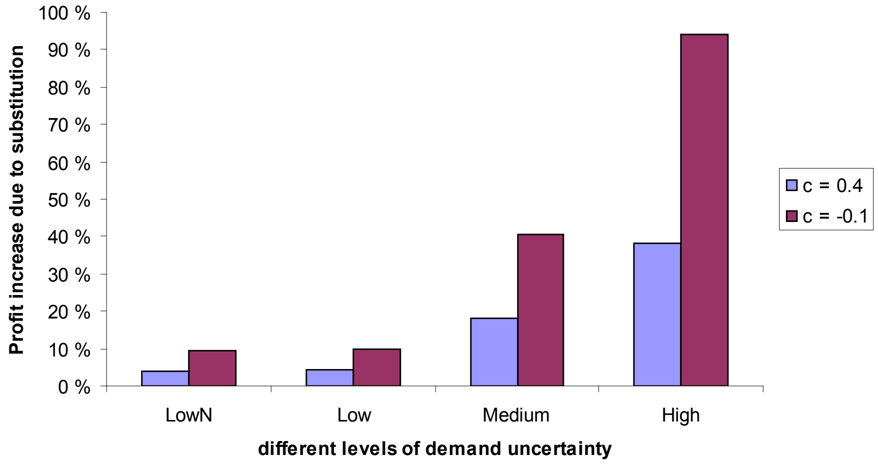

Figure 7 Rise in profit when increasing substitutability from 0 to 0.2, illustrated for correlations 0.4 and -0.1

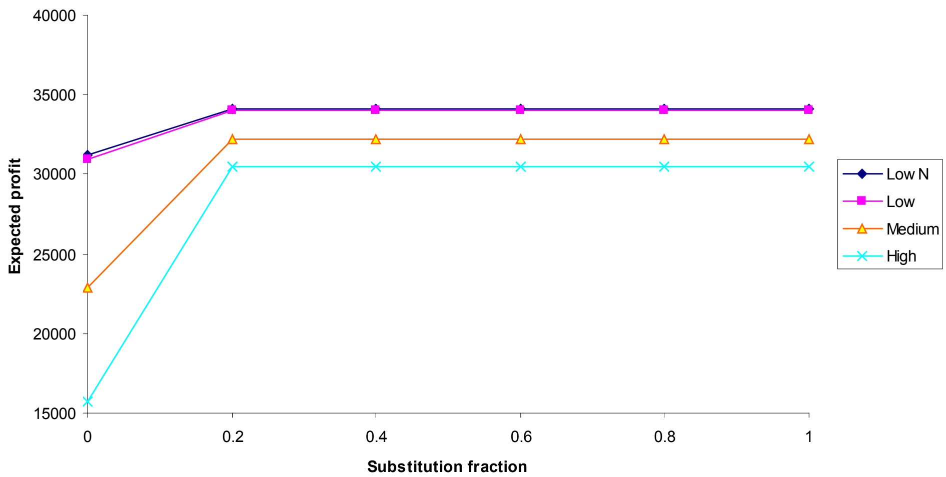

Figure 8 Expected profit versus substitution for different levels of demand uncertainty, 7-items, correlation = -0.1

Hence, we observe that to achieve a competitive variety strategy, the following apparent contradiction needs to be resolved: *offering items that are distinct but also similar.* In particular, the assortment items must be (i) distinct on some demand driver attributes (producing negative correlations), to achieve hedging by offering own 'competitors', *and* (ii) similar with regard to some other demand driver attributes, to enhance substitution and, hence, reduce the volatility of the portfolio (i.e. substitutable). We recognize that to achieve this, an attribute based product understanding is required, in particular how the product attributes create dependencies among the individual products. As such, it is rather obvious that complex and varying dependencies must be captured by the models. These patterns are largely lost when assuming average values on correlations and substitutability, and hence, whenever such simplifications are required by the chosen modelling approach.

As an illustration, consider a group of outdoor performance jackets for the summer season, from a leading brand name supplier. Assume that the aggregated level demand is rather stable but there is substantial uncertainty in individual item demands. This uncertainty stems from ephemeral fashion attributes, such as 'colour'. Assume that when one colour becomes popular the others will be unpopular (described by some negative correlation). Furthermore, assume that brand loyalty and specific technical attributes across the group make the items (modestly) substitutable. This latter reduces the volatility in sales and, hence, contributes to rise in profit. Modelling such effects require an attribute based understanding of products.

4. Conclusion

In this work we demonstrated that common simplifications applied in some analytical formulations — particularly on inter-item dependencies (correlation and substitution measures) *and* on the nature of demand distributions (more precisely, by replacing bi-modal with uni-modal distributions) — lead to reduced problem complexity and substantial differences in the portfolio structure and in the assortment profit. We notice, though, that the value of substitution is high and compensates, to some extent, for lack of information; however, *only given* that the 'true' substitution pattern is described by the model. We have found that actual expected profit can drop with as much as 30% when simplifications are applied. Further, we show that even when technology and supply chain flexibility allows for continuous information and production updates the underlying distributional and dependency assumptions used in the planning models are crucial.

Furthermore, we demonstrated that stochastic programming has an important role to play also in operational variety planning; not just by identifying optimal inventory levels, but also suggesting structural changes where appropriate, such as product line trimming. We show that the portfolio structure is indeed sensitive to the substitution pattern, and simplifying complex dependencies by taking average values, leads to significantly different assortment decision. Similar observations have also been seen in financial portfolio planning. This is clearly important to highlight when the negative effects of variety explosion are substantial for suppliers/retailers, and when the focus is changing from 'increasing variety to satisfy heterogeneous customer needs' to defining more efficient portfolio planning strategies.

The results obtained here are potentially important in assortment planning and could not have been achieved by the existing analytical formulations. The findings stem from the stochastic programming formulation, and from its very nature of allowing for complex distributional and dependency patterns. This latter is partially enabled by the use of scenario based distributions. Analytical solutions intend to avoid conclusions based on numerical calculations (and as such, the use of scenarios as well), but require major simplifications on dependencies. Although replacing heterogeneous demand correlation values by a single value is easier for practitioners to implement, decisions derived under such simplifications do not always reflect the assortment problem complexity. Being poorly fit for incorporating dependencies, some of the existing analytical models have limited potential in even detecting complex effects of mis-specifying these terms. Numerical examples constructed to illustrate the analytical models' behaviour, also reflect the same limitations.

We further strengthen the numerical models' usefulness in assortment planning by our qualitative and modelling related observation: To really leverage on the general insight of *substitution is more profitable under negatively correlated demands*, an attribute based view of the problem is necessary. Pairing items that are 'competing' for market acceptance (i.e. negatively correlated) but also 'similar' (i.e. substitutable), is far from straightforward. It requires knowledge and understanding of how the different product attributes (subjective trends or more concrete technical attributes) affect each other and the market demand. An attribute based view of the assortment problem is enabled in the applied SP approach; particularly, by the specific way inter-item dependencies are defined.

The stochastic programming approach analysed here is simple and handles real problems of substantial size; particularly fitted to treat the complex nature of demand uncertainty and dependencies observed in QR supply chains, such as fashion and sports apparel. The findings support earlier qualitative conclusions on the substitutable newsvendor problem, also under complex distributional assumptions. As a final remark, we mention that our substitutability and correlation values describing the dependencies are based on company data and qualitative understanding. As such, they are partially subjective and "incorrect" due to lack of sufficient trend/product/market knowledge. However, they are not completely off and are suited for illustrating the impact of mis-specifying, in our understanding, central terms.

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