Stability analysis of portfolio management with conditional value-at-risk

Michal Kaut and Stein W. Wallace and Hercules Vladimirou and Stavros Zenios

Norwegian University of Science and Technology, Trondheim, Norway
Lancaster University Management School, Lancaster, England
Center for Banking and Financial Research, University of Cyprus, Nicosia, Cyprus

in: Quantitative Finance. See also BibTeX entry below.

BibTeX:

```
@article{KautEA07,
  author = {Michal Kaut and Stein W. Wallace and Hercules Vladimirou and Stavros Zenios},
  title = {Stability analysis of portfolio management with conditional value-at-risk},
  journal = {Quantitative Finance},
  year = {2007},
  volume = {7},
  pages = {397-409},
  number = {4},
  doi = {10.1080/14697680701483222}
}
```

Stability analysis of portfolio management with conditional value-at-risk

Michal Kaut,† Hercules Vladimirou,‡ Stein W. Wallace,† and Stavros A. Zenios‡

January 22, 2010

Abstract

We examine the stability of a portfolio management model based on the conditional value-at-risk (CVaR) measure; the model controls risk exposure of international investment portfolios. We use a moment-matching method to generate discrete distributions (scenario sets) of asset returns and exchange rates so that their statistical properties match corresponding values estimated from historical data. First, we establish that the scenario generation procedure does not bias the results of the optimization program, and we determine the required number of scenarios to attain stable solutions. We then investigate the sensitivity of the CVaR model to mis-specifications in the statistics of stochastic parameters: mean, standard deviation, skewness, kurtosis, as well as correlations. The results are most sensitive to estimation errors in the means of the stochastic parameters (asset returns and currency exchange rates). Mis-specifications in the standard deviation, skewness and correlations of the random parameters also have considerable impact on the solutions. The effect of mis-specifications in the values of kurtosis, although less than that of the other statistics, is still not negligible.

1 Introduction

The conventional mean-variance approach, that constitutes the primary basis for portfolio selection, assumes that asset returns follow normal distributions and/or that the investor has a quadratic utility function. Despite the long and widespread use of the mean-variance method in portfolio management, its fundamental assumptions often do not hold in practice. The returns of many financial securities exhibit skewed and leptokurtic distributions. Derivatives, or securities with embedded options, have, by construction, highly skewed return distributions. Many other investments are exposed to multiple risk factors whose joint effect on portfolio returns often can not be modelled by a normal distribution.

Substantial research effort has been directed toward the development of models that properly capture asymmetries and dynamic effects in the observed behavior of asset returns. At the same time, alternative risk metrics have been sought. Such measures are concerned with other, or additional, characteristics of the return distribution (e.g., the tails) besides the variance, and can accommodate a wide range of investor priorities and regulatory requirements for risk management. Value-at-risk (VaR) has essentially attained the status of a de-facto standard in financial practice (see, e.g., Jorion [10]). VaR is defined as the maximal loss (or minimal return) of a portfolio over a specific time horizon at a specified confidence level; VaR corresponds to a percentile of the portfolio's loss (or return) distribution at a specified confidence level.

Despite its widespread popularity in recent years, VaR suffers from a number of theoretical and practical limitations. Although its calculation for a certain portfolio indicates that shortfall

†Molde University College, Servicebox 8, N-6405 Molde, Norway.

‡Center for Banking and Financial Research, School of Economics and Management, University of Cyprus, P.O. Box 20573, CY-1678 Nicosia, Cyprus.

returns, below `VaR`, will occur only with a prespecified likelihood, it provides no information on the extent of the distribution's tail which may be quite long; in such cases, the portfolio return may take substantially lower values than `VaR` and result in severe losses. More importantly, `VaR` is not a coherent risk measure in the sense defined by Artzner et al. [3]. It fails to reward diversification, as it is not subadditive; hence, the `VaR` of a diversified portfolio can be larger than the sum of the `VaR`s of its constituent asset components. Moreover, when the returns of assets are expressed in terms of discrete distributions (i.e., scenarios) `VaR` is a non-smooth and non-convex function of the portfolio positions and exhibits multiple local extrema (see, e.g., Rockafellar and Uryasev [15]). Incorporating such functions in mathematical programs is very difficult, thus making impractical the use of `VaR` in portfolio optimization models.

To overcome the deficiencies of `VaR`, suitable alternative risk metrics have been sought. Artzner et al. [3] discuss such metrics and specify the properties that sound risk measures should satisfy, which they characterize as coherent risk measures. A family of closely related risk metrics — termed as *expected shortfall*, *mean excess loss*, *tail VaR*, *conditional VaR* — have been suggested that quantify the mass in the tail of the distribution beyond `VaR`. Tasche [16] examines the properties of this family of measures; he shows that it characterizes the smallest coherent risk measures to dominate `VaR` and that it can incorporate higher moment effects. Acerbi and Tasche [1] show that the alternative definitions of these measures lead to the same results when applied to continuous loss distributions. They note that differences appear when the underlying distribution has discontinuities and they demonstrate that, in such cases, care must be exercised in the details of the definition to maintain the desired properties of coherence.

Rockafellar and Uryasev [15] introduced a definition of the *conditional value-at-risk* (`CVaR`) measure for general distributions, including discrete distributions that exhibit discontinuities, and showed that `CVaR` is a continuous and convex function of the portfolio positions. Most importantly, they showed that a `CVaR` optimization model can be formulated as a linear program in the case of discrete distributions of the stochastic input parameters. `CVaR` is defined as the conditional expectation of losses exceeding `VaR`; it is a coherent risk measure that quantifies the worst (lowest) portfolio returns below the respective `VaR`.[1] As `CVaR` is concerned with the tail of the distribution it is a suitable risk measure when the distribution is asymmetric and/or heavy-tailed.

As a result, `CVaR` models are seing increasing use in various financial management applications. For example, `CVaR` models have been suggested by Bogentoft et al. [4] for asset-liability management of pension funds, by Krokhmal et al. [13, 14] for hedge fund portfolios and by Anderson et al. [2] for credit risk optimization. Jobst and Zenios [9] showed that `CVaR` models are effective for modelling credit risk and accounting for default events in the tails. Topaloglou et al. [17, 18] applied `CVaR` models to international portfolio management problems to account for asymmetric and leptokurtic distributions of exchange rates and asset returns.

A specification of the distribution of stochastic parameters (asset returns) is a critical input for all portfolio management models. In parametric models, the multivariate distribution is specified by the values of key statistics that are usually estimated using historical data, analytical methods, analysts' forecasts and other methods. In nonparametric models, the distribution is usually represented in terms of a discrete set of plausible outcomes (scenarios) that are generated by simulation, bootstrapping historical data, or even subjective estimates in some cases. In all cases, the reliability of the model's results depends on the accuracy with which the postulated distribution approximates the true distribution of the random variables — which is never actually observable. Hence, the models are inevitably exposed to estimation errors. Consequently, it is important to understand the sensitivity of a model to mis-specifications of distributional characteristics, and to assess the relative effects that mis-specifications of various statistical properties have on the results. This can guide analysts in their choice among alternative estimation methods, as well as in the relative effort they invest to obtain robust estimates of the various model

[1]When defined in terms of portfolio returns, this risk measure is commonly referred to as *Return-at-Risk* (`RaR`).

inputs.

As CVaR is concerned with the tail of the portfolio's return distribution, models that employ this measure are expected to be sensitive to higher moments of the constituent assets' random returns. This paper aims specifically to study the stability of a CVaR portfolio management model with respect to changes in input specifications. In this respect, we follow previous studies on stability of mean-variance models.

Most notably, Kallberg and Ziemba [11], Chopra and Ziemba [6] and Broadie [5] examined the relative effects of estimation errors in the mean, variance and covariance of asset returns on mean-variance efficient portfolios. They found that the model results are most sensitive to mis-specifications in the means of asset returns. They reported that the impact of errors in the variance of asset returns was about an order of magnitude lower than that of errors in the means, while errors in covariance values had about half the impact of errors in the variance. Chopra and Ziemba found that the sensitivity of the model's results to estimation errors of statistical properties of asset returns varies with the level of risk aversion. Broadie pointed out that the impact of estimation errors on the mean-variance model increases with the number of securities included in the portfolio.

Here, we extend these studies as we similarly investigate the effects of mis-specifications in statistical properties — including higher moments — of asset returns on the results of a model based on the CVaR measure. As a test case we use a portfolio optimization model for international investments. The portfolio is exposed to market risk in multiple countries and to currency risk. We use discrete scenarios to model the uncertainty in asset returns and spot exchange rates. The scenarios are generated by the moment matching method of Høyland et al. [7] so that in the set of generated scenarios the random variables have statistical properties that match specific target values as determined from historical market data.

First, we define in-sample and out-of-sample stability and we demonstrate that the scenario generation procedure does not bias the results of the optimization model. That is, for sufficiently large scenario sets, the portfolio model produces stable solutions that are not dependent on the specific scenario sets (i.e., the results are stable with respect to sample). We then conduct extensive computational experiments to assess the effects on the model's results due to variations in the target statistics: mean, standard deviation, skewness, kurtosis and correlations of the random variables. We demonstrate that the CVaR model is indeed sensitive to the higher moments of the stochastic inputs. Moreover, we quantify the relative impact of mis-specifications in the various statistical properties of the inputs on the model's results.

The paper is organized as follows. Section 2 presents the CVaR model for international portfolio management that we use as a test case in this study. Section 3 describes the scenario generation method, the input data, and the tests to verify the stability of the optimization model with respect to the scenario generation procedure. In section 4 we describe the computational experiments involving mis-specifications in the statistical properties of stochastic input parameters and we present the effects of these errors on the model's results. Finally, section 5 concludes.

2 CVaR Model for International Portfolio Management

We test a CVaR model for international portfolio management. We view the problem from the perspective of a US investor who may construct a portfolio composed of domestic and foreign securities. Thus, we have a simple portfolio construction problem with a holding period of one month. To purchase foreign securities, the investor must first convert funds to the respective currency; the current spot exchange rates apply in the currency exchange transactions.

The asset set includes a stock index (Stk), a short-term (Bnd1) and a long-term (Bnd7) government bond index in each of four countries: United States (USA), United Kingdom (UK), Germany (Ger) and Japan (Jap). The values of the assets and the exchange rates at the end of the holding period are uncertain; their joint distribution is modelled in terms of a scenario set (i.e., a set of discrete outcomes with associated probabilities).[2] At the end of the holding period we compute the scenario-dependent value of each investment using its projected price under the respective scenario. The USD-equivalent value is determined by applying the estimate of the appropriate spot exchange rate to USD at the end of the period under the same scenario.

The portfolio is exposed to market risk in the various countries, as well as to currency risk. To (partly) hedge the currency risk, the investor may enter into forward currency exchange contracts. The monetary amounts (in USD) of forward contracts are decided at the time of portfolio selection, but the currency exchanges are executed at the end of the holding period.

We define the following notation:

User-specified parameters:

α	confidence (percentile) level for `VaR` and `CVaR`
ϑ	minimal allowable `CVaR` of portfolio returns

Sets and indices:

$\boldsymbol{M}$	set of markets (synonymously, countries, currencies)
$\ell \in \boldsymbol{M}$	index of investor's base (reference) currency in the set of currencies
$\boldsymbol{M}_f$	set of foreign markets; $\boldsymbol{M}_f = \boldsymbol{M} \setminus \{\ell\}$
$\boldsymbol{I}_j$	set of available asset classes (stock and bond indices) in market $j \in \boldsymbol{M}$
$\boldsymbol{S}$	set of scenarios: $\boldsymbol{S} = \{1, \ldots, S\}$

Deterministic input data:

c_ℓ	amount of initially available cash in base currency ℓ, ($c_\ell = 100$)
π^0_{ij}	current market price of asset $i \in \boldsymbol{I}_j,\ j \in \boldsymbol{M}$; in units of local currency j
γ_{ij}	transaction cost rate for purchases of asset $i \in \boldsymbol{I}_j,\ j \in \boldsymbol{M}$, ($\gamma_{\texttt{Stk},j} = 0.001\,,\ \forall j \in \boldsymbol{M}\ ;\ \gamma_{\texttt{Bnd1},j} = \gamma_{\texttt{Bnd7},j} = 0.0005\,,\ \forall j \in \boldsymbol{M}$)
λ	transaction cost rate for spot currency exchanges, ($\lambda = 0.0001$)
e^0_j	current spot exchange rate of currency $j \in \boldsymbol{M}$
φ_j	current one-moth forward exchange rate of currency $j \in \boldsymbol{M}$ (i.e., the market-quoted rate for a currency exchange to be executed at the end of the holding period)

Scenario dependent data:

S	number of scenarios, $S = \lvert\boldsymbol{S}\rvert$
p_s	probability of scenario $s \in \boldsymbol{S}$ — in our tests, scenarios are equiprobable (i.e., $p_s = 1/\mathrm{S}$)
π^s_{ij}	price of asset $i \in \boldsymbol{I}_j,\ j \in \boldsymbol{M}$ at the end of the holding period under scenario $s \in \boldsymbol{S}$; in units of local currency j
e^s_j	spot exchange rate of currency $j \in \boldsymbol{M}$ at the end of the holding period under scenario $s \in \boldsymbol{S}$

Decision variables:

x_{ij}	number of assets $i \in \boldsymbol{I}_m,\ j \in \boldsymbol{M}$ in the portfolio, in units of face value
f_j	amount of base currency collected from sale of currency $j \in \boldsymbol{M}_f$ in the forward market (i.e., amount of forward contract, in units of the base currency)

Auxiliary variables:

[2] In this model instance we have 15 random variables: the returns of the 12 indices (3 for each of the 4 countries) during the holding period, and the exchange rates of the 3 foreign currencies to USD at the end of the holding period.

v^s total value of the portfolio at the end of the holding period under scenario $s \in \boldsymbol{S}$, in units of the base currency

r^s portfolio return under scenario $s \in \boldsymbol{S}$

z variable in definition of `CVaR` — equals `VaR` at the optimal solution

y_s return shortfall below `VaR` under scenario $s \in \boldsymbol{S}$, $y_s = [\,0, z - r^s\,]^+$

All exchange rates are expressed as the equivalent amount of the base currency for one unit of the foreign currency. Obviously, the exchange rate of the base currency to itself is trivially equal to one $e_\ell^0 = e_\ell^s = 1,\ \forall\, s \in \boldsymbol{S}$.

We formulate the international portfolio selection model as follows:

$$\texttt{maximize} \quad \sum_{s \in \boldsymbol{S}} p_s r^s \tag{1a}$$

$$\texttt{s.t.} \quad c_\ell = \sum_{i \in \boldsymbol{I}_\ell} x_{i\ell} \pi_{i\ell}^0 (1 + \gamma_{i\ell}) + \sum_{j \in \boldsymbol{M}_f} e_j^0 (1 + \lambda) \left(\sum_{i \in \boldsymbol{I}_j} x_{ij} \pi_{ij}^0 (1 + \gamma_{ij}) \right) \tag{1b}$$

$$v^s = \sum_{i \in \boldsymbol{I}_\ell} x_{i\ell} \pi_{i\ell}^s + \sum_{j \in \boldsymbol{M}_f} \left\{ f_j + e_j^s \left(\sum_{i \in \boldsymbol{I}_j} x_{ij} \pi_{ij}^s - \frac{f_j}{\varphi_j} \right) \right\}, \qquad \forall\, s \in \boldsymbol{S} \tag{1c}$$

$$r^s = (v^s - c_\ell)/c_\ell\,, \qquad \forall\, s \in \boldsymbol{S} \tag{1d}$$

$$y_s \geq z - r^s\,, \qquad \forall\, s \in \boldsymbol{S} \tag{1e}$$

$$z - \frac{1}{1-\alpha} \sum_{s \in \boldsymbol{S}} p_s y_s \geq \vartheta \tag{1f}$$

$$y_s \geq 0\,, \qquad \forall\, s \in \boldsymbol{S} \tag{1g}$$

$$f_j \geq 0\,,\ x_{ij} \geq 0\,, \qquad \forall\, j \in \boldsymbol{M}\,, \forall\, i \in \boldsymbol{I}_j \tag{1h}$$

This is a simplified version of the international portfolio management model in Topaloglou et al. [17]. The model in that paper accounts for an initial portfolio — including cash and asset positions in any currency — and determines transactions — asset sales and purchases, as well as spot currency exchanges — so as to obtain a revised portfolio. A multi-stage extension of that model to address dynamic international portfolio management problems is developed in Topaloglou et al. [18]. Here, we consider a simple portfolio construction model that selects a portfolio starting with an initial cash endowment in the base currency only.

The objective function (1a) maximizes the expected portfolio return over the holding period. Equation (1b) is the budget constraint; it indicates that the cost for the purchase of domestic and foreign assets is covered by the available cash (c_ℓ). Linear transaction costs (γ_{ij}) are charged for asset purchases (x_{ij}); a linear transaction cost (λ) is also charged for spot currency exchanges that are effected in order to purchase foreign assets. Note that the entire budget is allocated to the available assets; a simple extension of the model can allow investments in money market accounts in the various currencies.

Equations (1c) determine the scenario-dependent values of the portfolio (v^s), in units of the base currency, at the end of the holding period. These valuation equations account for the revenues from the liquidation of all portfolio positions at the end of the holding period at the projected asset prices (π_{ij}^s) for the corresponding scenario. The contribution of foreign investments to the total value of the portfolio accounts for the settlement of any outstanding forward contracts (f_j). The residual amount in a foreign currency is valued in terms of the base currency by using the projected spot exchange rates (e_j^s) at the end of the holding period. Equations (1d) determine the return of the portfolio under each scenario.

Constraints (1e) and (1g) determine the excess shortfall returns, beyond VaR, under each scenario. Constraint (1f) imposes a minimal allowable value (ϑ) on the CVaR of portfolio returns over the holding period at the $(1-\alpha)100^{\text{th}}$ percentile. At the optimal solution, the variable z is equal to the VaR at the same percentile — when constraint (1f) is active, which is always the case in this model. The constraints in (1h) disallow short positions.

The linear programming formulation of CVaR models when the stochastic inputs follow a discrete distribution is due to Rockafellar and Uryasev [15]. The model here maximizes the expected portfolio return while constraining the CVaR value of portfolio returns; with constraint (1f) the expected excess losses in the tail of the distribution, beyond VaR, are bounded by the parameter ϑ. Financial optimization models with CVaR constraints are reported, for example, in [2, 4, 13, 14]. Alternatively, we could have opted to maximize the CVaR of portfolio returns and impose a minimal target on expected return, as is done in Topaloglou et al. [17]. We chose this formulation as it is more natural to interpret the impact of estimation errors in stochastic inputs on the expected portfolio return, rather than on the value of a risk measure.

As it was shown in Topaloglou et al. [17, 18] the monthly variations of exchange rates — also the returns of several stock indices — exhibit skewed and fat-tailed distributions. The use of the CVaR metric is appropriate in the context of the international portfolio management model, as it can accommodate the skewed and leptokurtic distributions of the stochastic inputs (see Table 3 in the Appendix). As we noted earlier, many other portfolio management models involve securities with asymmetric and leptokurtic return distributions, for which a CVaR model would be suitable.

3 Scenario generation

We used the method of Høyland et al. [7] to generate scenarios of asset returns and spot currency exchange rates. The method generates a set of discrete scenarios for the random variables so that the first four moments of the marginal distributions (mean, standard deviation, skewness and kurtosis), as well as the correlation coefficients match specified targets. We estimate the target values for these statistics from historical data. However, this is not a prerequisite for the scenario generation procedure. We could, as easily, use subjective estimates for the target statistics, as well as target values determined with alternative estimation procedures.

The moment-matching method allows full control of the moments when generating scenarios. This capability is essential for the purposes of this study. To investigate the impacts on the model of variations in the values of some moment of the random variables, we need a procedure that can generate scenarios effected only in terms of the moment studied, while keeping all other statistical properties of the stochastic inputs unchanged. The moment-matching method provides this capability.

3.1 Data

The data for the stock indices were obtained from the Morgan Stanley Capital International, Inc. database (www.mscidata.com). The data for the bond indices and the currency exchange rates were collected from DataStream. All time series have a monthly time-step and cover the period from January 1990 to April 2001 (i.e., a total of 136 monthly observations). The statistical properties of these data series are reported in Tables 3 and 4 in the Appendix.

3.2 Assessment of the scenario generation method

In Section 4, we investigate the behavior of the CVaR model with respect to the number of scenarios and with respect to mis-specifications in the statistical properties of stochastic inputs. To ensure the reliability of the results, however, we must first show that the scenario generation method used does not influence the results by causing instability of the solutions. That is, if the solutions change for different scenario sets then the results of section 4 would be suspect.

Ideally, we would like to determine that the scenario generation procedure can effectively produce robust solutions with respect to the true distribution of the random variables. This is not an attainable goal as the true distribution is not observable. Hence, we assess the scenario generation method in terms of its ability to closely approximate a benchmark distribution, and the stability of the results with respect to the benchmark. It is important that the benchmark distribution is provided exogenously, that is, it is not generated by the same method which we are testing.

We use as benchmark a discrete distribution (scenario set) generated by a method based on principal component analysis as described in Topaloglou et al. [17]. The benchmark distribution has $15,000$ scenarios that jointly depict the co-variation of the 15 random variables in the international portfolio management problem. We note that the scenarios of the benchmark tree are not equiprobable. From the benchmark scenario set, we compute the moments and correlations of the random variables. We use these values as the target statistics to match with the scenario generation procedure.

First, we verify that moments of the random variables in the scenarios sets that we generate match the target values. We also check that the generated scenario sets reproduce other distributional characteristics (e.g., the entire marginal distributions).

Matching marginal distributions

The easiest to check is a match of the marginal distributions. We generated scenario sets ranging in size from 250 to $5,000$ scenarios. For each set we determined the marginal distributions of the random variables from generated scenario sets and compared them to the corresponding distributions from the benchmark. The comparison in the case of the US stock index (`Stk.USA`) is depicted in Figure 1. The reproduction of the marginal distributions of the remaining random variables is quite similar.

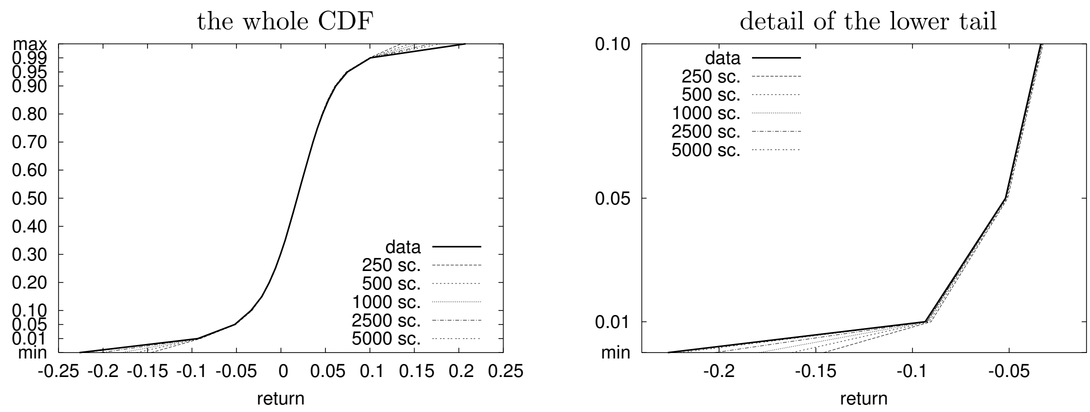

Figure 1: Match of the distribution function for the US stock index. Comparison of the distribution function for the monthly returns of the US stock index in the benchmark (data) and in generated scenarios sets of different size. *The scale of the vertical axis is not linear, the two outer intervals are prolonged.*

We observe that even with moderate-size scenario sets (> 250 scenarios) we can closely reproduce the marginal distributions from the 1st to the 99th percentile. At the extreme tails the distribution is not as accurately matched unless a sufficiently large number of scenarios is generated. This is understandable, as we should expect more samples in the tails as the number of scenarios increases.

The desired degree of matching the distributions depends on the decision model in which the scenarios will be used. For example, if we are to apply a mean-variance model then the accuracy of match at the tails will not make any difference, as long as the first two marginal moments and the correlations are matched. A close match of the tails becomes relevant for the `CVaR` model which is concerned with the tail of the portfolio's return distribution.

The match of the marginal distributions of the random variables is illustrative. Yet, it is not sufficient, even though the generated scenario sets also match the desired correlations as well. We need to establish that the portfolio optimization model produces stable results regardless of the specific scenario set generated in any given run — i.e., that it is stable with respect to sample. Evidently, scenario sets of sufficiently large size are needed to ensure such stability. We need to test jointly the scenario generation method and the optimization model in order to verify that the scenario generation method does not cause instability of the solutions.

Joint stability test of scenario generation and the `CVaR` model

We generate 25 scenario sets of a given size, each matching the moments and correlations of the benchmark distribution. We solve the optimization model with each scenario set and record the optimal expected portfolio return. The confidence level in all tests is $\alpha = 0.95$; thus, `CVaR` is the expected return for the 5% worst scenarios. The bound on `CVaR` is $\vartheta = -1\%$. As the constraint (1f) is always active, the `CVaR` of portfolio returns is always at its minimal value (-1%) at the optimal solution; thus, the expected losses over the 5% worst scenarios are 1%.

As stochastic programs tend to have multiple optimal or near-optimal solutions, we study the stability in terms of the optimal value; we do not compare the optimal portfolio compositions. We then simulate all the solutions on the benchmark distribution and record the out-of-sample values of both the expected return and `CVaR`.

As in [12] we examine two types of stability:

In-sample stability: The solutions should not vary across scenario sets of the same size. We examine the in-sample variation of the optimal values (expected return) across the 25 scenario sets of a given size; ideally these should be equal.

Out-of-sample stability: We examine the variation of the expected portfolio returns and `CVaR` values obtained when the solutions are simulated on the benchmark distribution. These out-of-sample values should ideally be equal for all scenario sets. They should also be equal to the in-sample values.

The two notions of stability are not equivalent. We can have in-sample stability without out-of-sample stability. Consider, for example, a case in which all the scenario sets are identical but incorrect in comparison to the benchmark. On the other hand, we can have alternative scenario sets for which the model yields the same optimal solution. Then the in-sample objective values could differ for different scenario sets, but the out-of-sample values would be equal.

Verifying out-of-sample stability in terms of a benchmark indicates that the model yields robust solutions that do not vary with respect to sample. This is essential in this study as we need to ensure that the variations in the solutions that are observed in the tests of the next section are caused by the variations in the statistics of the inputs, and not by instability with respect to sample. Hence, the purpose of our tests is to assess whether we can achieve both types of stability.

Results of the tests are depicted in Figure 2. We see that as we increase the number of scenarios to around $5,000$ we indeed achieve both in-sample and out-of-sample stability. Thus, the scenario generation method is effective, in the sense that it does not cause instability in the solutions of the `CVaR` model.

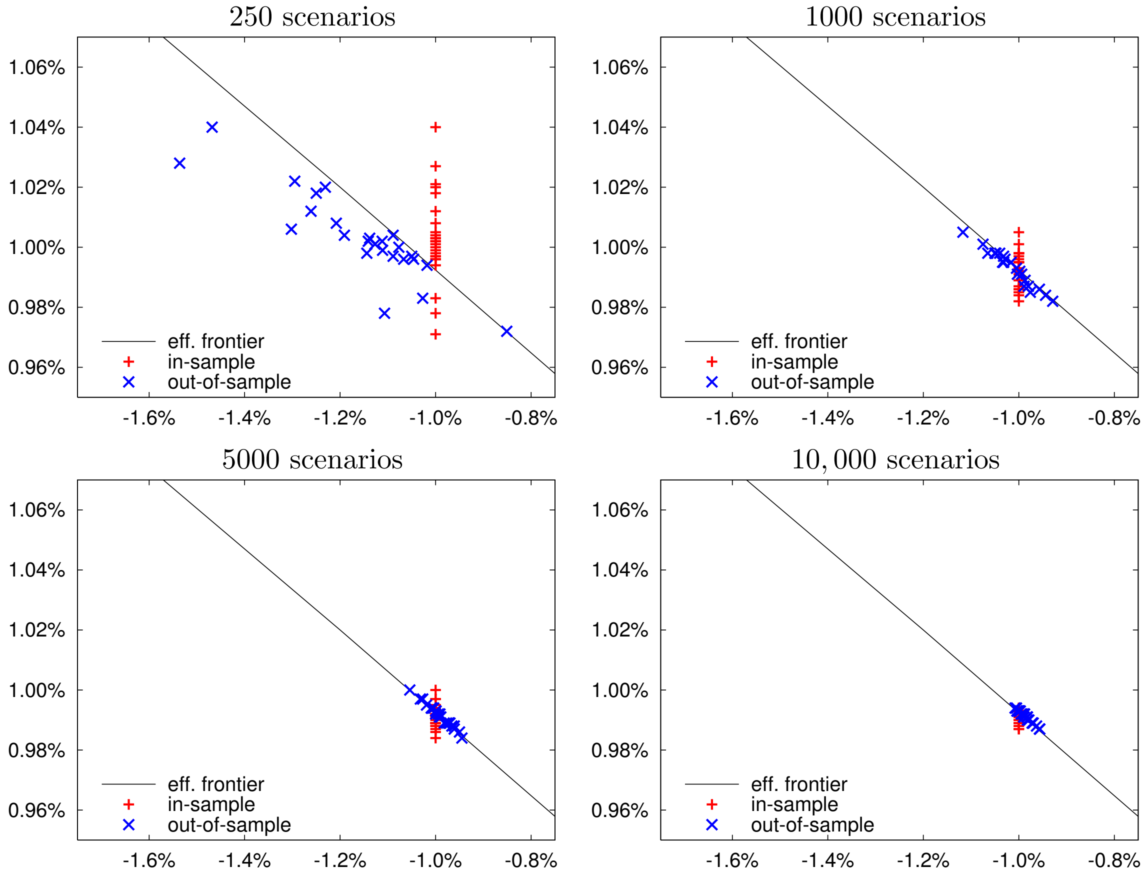

Figure 2: Stability of the `CVaR` model with respect to the exogenous benchmark. The horizontal axis shows the CVaR value, the vertical axis the expected portfolio return (monthly).

4 Sensitivity tests of the `CVaR` model

This section studies the sensitivity of the `CVaR` model with respect to mis-specifications in the statistics of stochastic parameters. Again we need a benchmark as a reference. Here we calibrate the scenario generation method using the statistical properties estimated from historical market data, as reported in Tables 3 and 4; all scenario sets in the following tests are generated so as to match these target statistics. First, we generate a benchmark with 20,000 scenarios. This scenario set is sufficiently large (according to the findings of the previous section), to ensure both in-sample and out-of-sample stability of the solutions, while it is still easily solvable so as to trace the reference efficient frontier. The efficient frontier, depicting the tradeoff between expected portfolio return and the `CVaR` risk metric, is obtained by repeatedly solving the parametric optimization model for different allowable limits ϑ on `CVaR`.

To interpret the results of the tests, we must understand the source of differences between the in-sample and out-of-sample expected portfolio return of a given portfolio. If the portfolio is invested solely in domestic assets, the expected return would depend only on the portfolio composition and the means (expected values) of asset returns. The contribution of a foreign asset, however, on the portfolio's return depends on the product of the asset return (in its local currency) and the change of the exchange rate to the reference currency. When foreign investments are present in a portfolio, the return depends on products of random variables; hence, the expected portfolio return depends not only on the means of the random variables, but also on their covariances. Thus, for a given portfolio, the in-sample and the out-of-sample expected portfolio returns would be equal only if the random variables have the same means, standard deviations, and correlations in the respective scenario sets (i.e., the test set and the benchmark). This condition is satisfied by construction in our scenario generation method as the random variables have matching moments and correlations in the benchmark and in the test sets. Hence, a portfolio has the same in-sample and out-of-sample expected return, but its `CVaR` value is different when it is simulated on the benchmark scenario set in comparison to its value on a test set.

4.1 Finding a sufficient number of scenarios

Before proceeding to the sensitivity tests, we verify that we employ sufficiently large scenario sets in our tests to ensure stability with respect to sample. That is, we must ensure that variations observed in the model's results stem from changes in the statistical properties of the stochastic inputs and not from insufficiency of the scenario test sets. In the tests of section 3.2 we found that at least $5,000$ scenarios were needed to attain both in-sample and out-of-sample stability to an acceptable level. As the benchmark is now different we repeat the same tests here; the statistics of both the benchmark as well as the test sets in this section were estimated from time series of market data. The results of the tests are summarized in Figure 3. Again, we observe that we need at least $5,000$ scenarios to ensure adequate stability of the `CVaR` model.

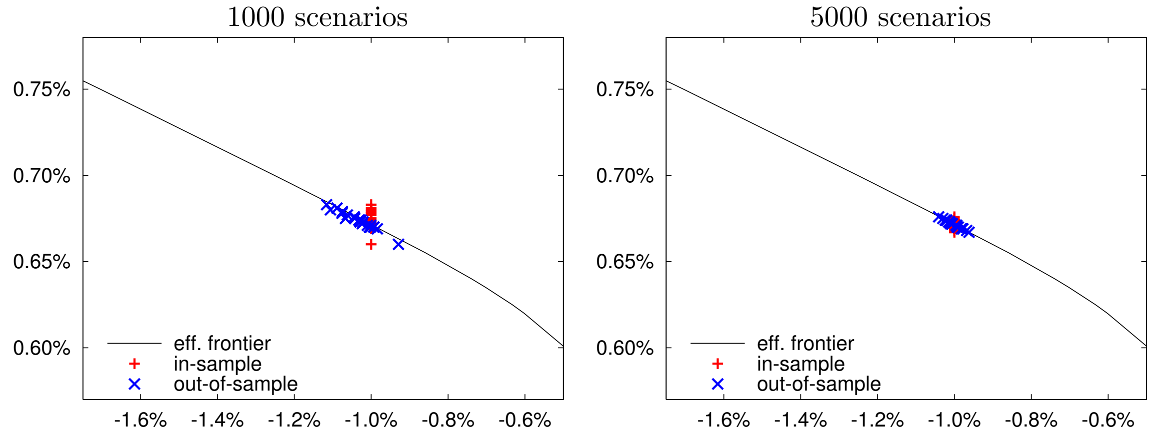

Figure 3: Stability of `CVaR` model with respect to the number of scenarios. The horizontal axis shows the CVaR value, the vertical axis the expected portfolio return (monthly).

Some comments on the figures:

- The in-sample results always lie on a vertical line, as the `CVaR` value is always equal to its minimal allowable limit ϑ at the optimal solution. The range of this line indicates the in-sample variation of expected return with respect to scenario sets of a given size.
- As the in-sample values are computed on the respective scenario sets — and not the benchmark — they can cross the reference efficient frontier that is generated using the benchmark.
- Because the random variables have the same moments and correlations in the test sets and in the benchmark, the in-sample and the out-of-sample expected portfolio returns are the same for a given portfolio, as we explained at the start of this section. Only the `CVaR` values of a portfolio change when it is simulated on the benchmark scenario set.

Table 1 presents measures of variation of expected portfolio returns for tests using scenario sets of increasing size. Variations in expected return decrease monotonically with increasing number of scenarios; this is a helpful consistency check.

# of scenarios	250	500	1000	2500	5000	10000
standard deviation	0.010%	0.007%	0.005%	0.002%	0.002%	0.001%
range ($\texttt{max} - \texttt{min}$)	0.034%	0.026%	0.023%	0.009%	0.006%	0.003%

Table 1: Standard deviation and range of out-of-sample expected portfolio returns.

4.2 Effects of mis-specifications in statistical properties

In this section we test the sensitivity of the CVaR model with respect to mis-specifications in the statistical properties of the stochastic inputs. Controlled errors are systematically introduced to the target statistics (first four moments and correlations) of the random variables at the scenario generation phase. Multiple scenario test sets are then generated to match the perturbed statistical properties. Similar tests for the mean-variance model are presented in Kallberg and Ziemba [11] and Chopra and Ziemba [6].

We quantify the induced errors by means of the following approach. We compute the moments and correlations of the random variables based on subsets of our data set, using a moving time window of $1/2$ the size of the available time series. Thus, we obtain a series of plausible estimates for the moments and correlations of the random variables. For each statistic, we take the interval from the minimal to the maximal estimated value which we call the *variation interval* for the corresponding statistic. These variation intervals for moments and correlations are reported in Tables 5 and 6 in the Appendix. We term the value of the respective statistic, calculated on the basis of the entire data set, the *true value*.

We define a δ-percent error in a statistical property as

$$\textit{true value} + \varepsilon \, \tfrac{\delta}{100} \, \texttt{length}(\textit{variation interval}) \,, \tag{2}$$

where ε is a random number from the uniform distribution on the interval $[-1, 1]$. With this definition, the average absolute error is

$$\tfrac{1}{2} \tfrac{\delta}{100} \, \texttt{length}(\textit{variation interval}).$$

Note that this is different from the corresponding definition in Chopra and Ziemba [6]. There, the δ-percent error was defined as $\textit{true\_value} \left(1 + \varepsilon \, \tfrac{\delta}{100}\right)$, $\varepsilon \in \mathcal{N}(0, 1)$. If we have a statistic (e.g., skewness or correlation) with a *true value* equal to zero, then this statistic would never be changed if we introduced errors using the approach of Chopra and Ziemba; variations of statistics with very small values would also be very small. For this reason, we chose to calculate controlled errors by means of (2).

A potential problem when introducing random errors to statistical properties is that we may specify a property, or a combination of properties, that is not feasible. For example, we may end up with specifications that may violate the condition $\textit{kurt} > 1 + \textit{skew}^2$, or we may specify a correlation matrix that is not positive definite. When this happens, we simply discard these particular specifications.

To test the impact of mis-specifications in each statistic we generate 100 scenario sets (with $5,000$ scenarios each) by randomly varying the value of the statistic. Every test proceeds as follows:

i. For each random variable, perturb the selected statistic using (2).
ii. Generate a scenario set, matching the perturbed statistical properties.
iii. Solve the portfolio optimization model and record the expected portfolio return and the value of CVaR at the optimal solution.
iv. Simulate the solution on the benchmark — which was generated with unperturbed statistics — estimating the expected portfolio return and the value of CVaR.

Results of the sensitivity tests

We ran tests for 10% and 25% errors (i.e., for $\delta = 0.10$ and 0.25). In the case of errors in marginal moments, we never obtained an infeasible specification. In the case of 25% errors in correlations, however, many of the generated correlation matrices were not positive definite, and were discarded and replaced. The discarded cases resulted from samples that introduced the larger levels of errors, i.e., $\varepsilon \approx \pm 1$. As the large error instances were discarded, the effective errors in correlations in this case are somewhat smaller.

The results of the tests with parameter settings $\alpha = 0.95$ and $\vartheta = -1.0\%$ are shown in Figures 4, and 5 for 10% and 25% errors, respectively. We observe that the larger levels of error, in any statistic, have a discernibly higher impact on the expected portfolio return. Estimation errors in the means clearly exhibit the highest impact on the solutions, followed by errors in standard deviations.

To examine the effects of estimation errors at different levels of risk aversion, we repeated the tests at different levels of the parameters α and ϑ; the combination of these parameters relates to the level of risk aversion. Increasing values of the percentile level α refer to more extreme tails of the return distribution. The parameter ϑ controls the allowable mass in the tail of the distribution; thus, lower values of this parameter (in absolute terms) constrain more tightly the size of the tail (beyond the percentile level specified by α), and reflect higher risk aversion. Results of the tests are summarized in Table 2.

	α	ϑ	benchmark expected return	Range of out-of-sample expected return estimates caused by estimation errors in respective statistics: mean	stdev	skew	kurt	corr
10% error	95%	-0.5%	0.601%	0.065%	0.035%	0.031%	0.024%	0.022%
	95%	-2.0%	0.782%	0.045%	0.025%	0.022%	0.020%	0.027%
	99%	-2.0%	0.707%	0.060%	0.034%	0.030%	0.022%	0.031%
25% error	95%	-0.5%	0.601%	0.255%	0.073%	0.058%	0.039%	0.031%
	95%	-2.0%	0.782%	0.247%	0.058%	0.028%	0.016%	0.039%
	99%	-2.0%	0.707%	0.247%	0.057%	0.034%	0.036%	0.043%

Table 2: Sample ranges ($\texttt{max} - \texttt{min}$) of out-of-sample expected (monthly) portfolio return estimates caused by estimation errors in statistics of the stochastic inputs.

Table 2 summarizes the impacts of estimation errors in statistics of the stochastic inputs to the `CVaR` model. It reports the sample ranges of out-of-sample expected (monthly) return estimates when errors were introduced to the respective statistics. For all statistics, the impact of errors increases with the level of the error; the larger rate of increase results from errors in the means, followed by errors in the standard deviations. We assess the impacts of estimation errors in statistics of the stochastic inputs in terms of the variation they induce in the out-of-sample estimates of expected portfolio return.

We summarize our observations on the results as follows:

- In all tests, the `CVaR` model is most sensitive to errors in the means of the stochastic inputs. The mean is by far the most important statistical property to estimate accurately, as errors in the means of the random variables have the most significant impact on the results — about 2 to 10 times higher than the effect of errors in any other statistical property.
- After errors in the means, mis-specifications of standard deviations have the next most important impact on the out-of-sample estimates of expected portfolio return — about 2

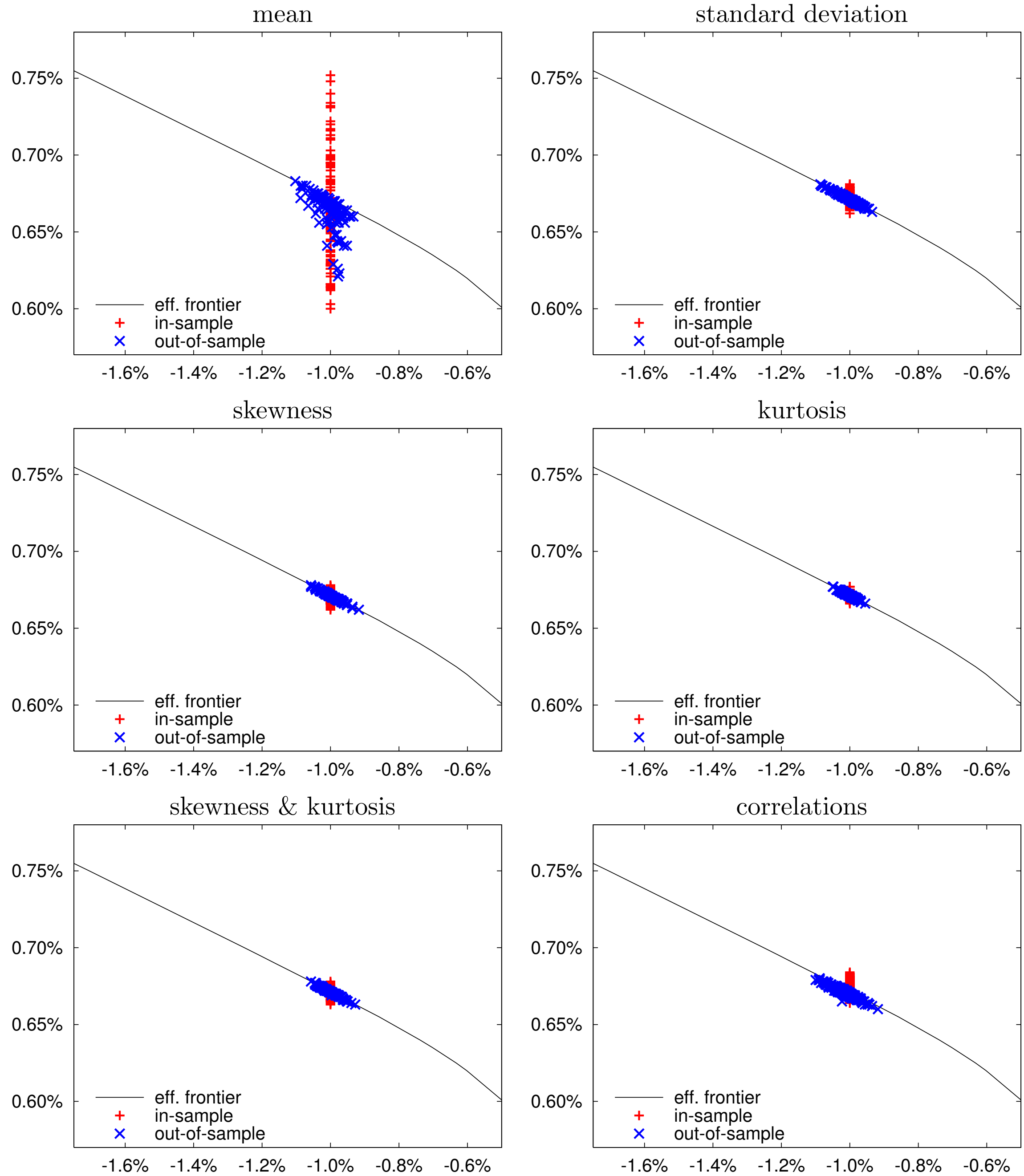

Figure 4: Stability of CVaR model with parameters $\alpha = 95\%$, $\vartheta = -1\%$; 10% estimation errors in the respective statistics. The horizontal axis shows the CVaR value, the vertical axis the expected monthly return.

to 5 times lower than the impacts of the means. Next is the impact of estimation errors in values of skewness — about 2 to 9 times lower than the impacts of the means — followed by the impact of errors in the estimates of kurtosis — about 3 to 12 times lower compared to the impacts of the means. The effects of estimation errors in correlations are between those of the standard deviations and skewness.[3]

[3]We note that the effects of mis-specifications in correlation values at 25% errors are underestimated. In these tests, samples with the larger levels of errors in correlations yielded non-positive definite correlation matrices and were discarded. Hence, the effective estimation errors in this case are somewhat lower, and the corresponding

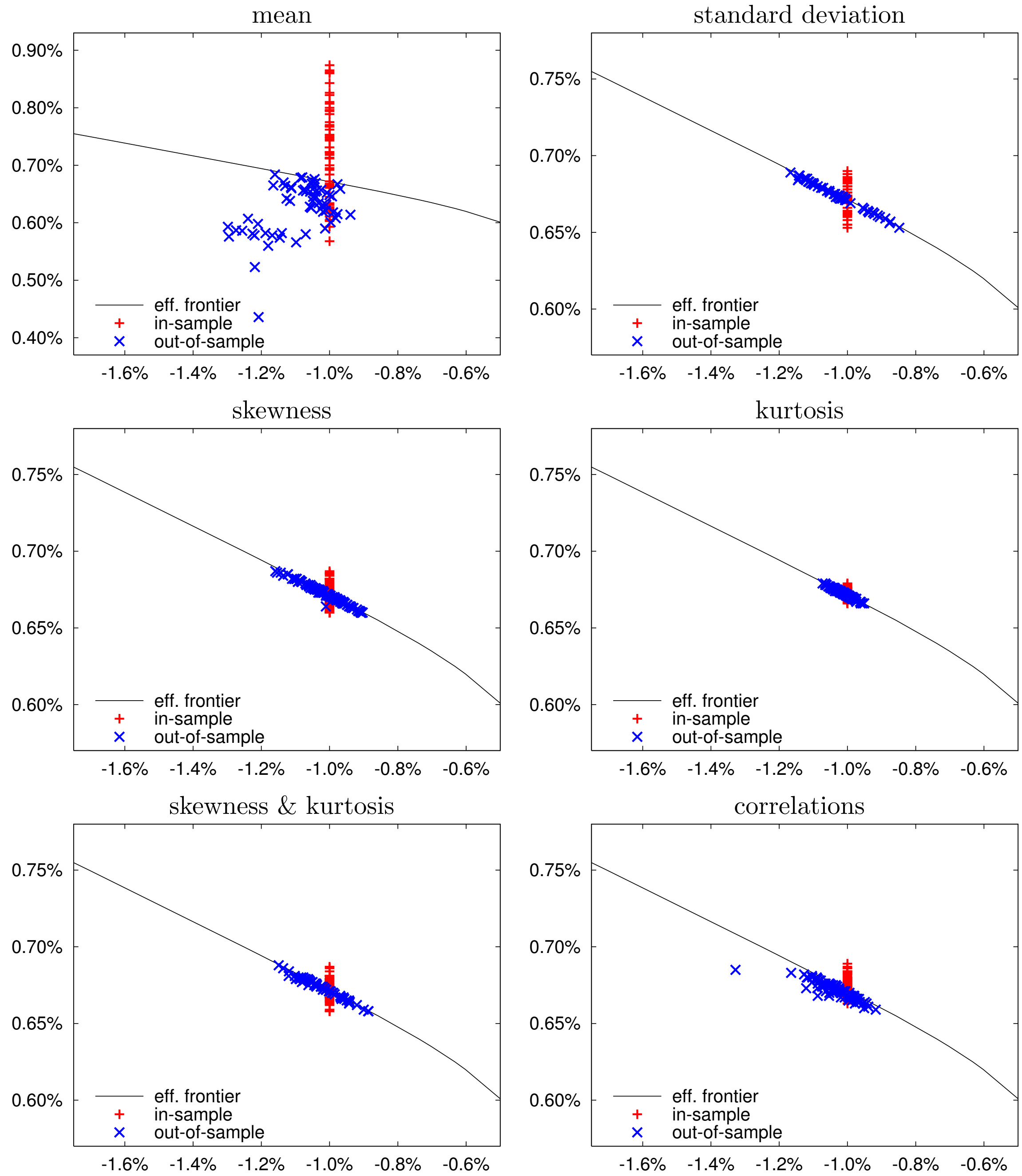

Figure 5: Stability of `CVaR` model with parameters $\alpha = 95\%$, $\vartheta = -1\%$; 25% estimation errors in the respective statistics. The horizontal axis shows the CVaR value, the vertical axis the expected monthly return. *The graph for errors in means has a bigger scale on the vertical axis.*

- Errors in the values of kurtosis have a detectable and non-negligible influence on the results of the `CVaR` model, though their impact is lower in comparison to that of the other statistics.
- For all marginal moments, the effect of estimation errors seems to increase for more risk averse settings of the model's parameters, i.e., for smaller allowable tails (as controlled by the parameter ϑ).
- The mean and the correlations are the only statistics whose mis-specifications result in

effects are underestimated.

portfolios that deviate significantly from the efficient frontier. Errors — especially when they are relatively small — in the other statistics seem to result in portfolios with different `CVaR` values, which are, however, still close to the efficient frontier.

The results demonstrate that estimation errors in higher-order moments of stochastic inputs do indeed affect in measurable ways portfolio management models that use the `CVaR` risk measure. The results imply that it pays to devote care and effort so as to accurately estimate the values of higher-order moments when employing risk measures concerned with the tails of the return distribution in portfolio management models.

5 Conclusions

We tested a risk management model for international portfolios based on the `CVaR` risk metric. We employed a scenario generation procedure based on principles of moment matching. We showed that this scenario generation method is effective and "unbiased", in the sense that it can closely reproduce the characteristics of a desired distribution and it leads to stable solutions of the portfolio optimization model.

We investigated the sensitivity of the `CVaR` model with respect to errors introduced to the statistical properties of stochastic inputs, as represented in discrete scenario sets. The statistical properties investigated included the first four marginal moments and the correlations of the random variables (assets returns and spot currency exchange rates). The tests quantify the relative effects of errors in these statistics on the model's results. The results confirm that the mean value of the random variables is the most important statistic to accurately estimate; the `CVaR` model exhibits high sensitivity to mis-specifications of the means. But, unlike the mean-variance model, the `CVaR` model shows sensitivity to errors in the estimates of higher-order moments as well. Errors in the standard deviation, correlations and skewness of the random variables have considerable impact on the model's results, in this order of importance. Estimation errors in the values of kurtosis have lesser, yet non-negligible, effects.

When assessing the potential effects of estimation errors in statistical properties, we should have a sense of the magnitude of such errors in practice. Much more care is exercised in generating reliable estimates of the means, variances and covariances of random financial variables, and more effective tools are available for their estimation, in comparison to higher-order moments. This is because the mean-variance model continues to be the primary paradigm for portfolio management, and because the importance of the first two moments is well understood. The need for accurate estimates of higher-order moments is often overlooked as their potential impact in portfolio management models is not as well understood and appreciated. This study sheds some light in this respect, by indicating the relative importance of accurate estimates of higher-order moments for random variables in risk management models that employ risk measures tailored to control the tails of the portfolio's return distribution.

References

[1] C. Acerbi and D. Tasche. On the coherence of expected shortfall. *Journal of Banking and Finance*, 27(6):1487–1503, 2002.

[2] F. Anderson, H. Mausser, D. Rosen, and S. Uryasev. Credit risk optimization with conditional value-at-risk criterion. *Mathematical Programming*, 89(2):273–291, 2001.

[3] P. Artzner, F. Delbaen, J.M. Eber, and D. Heath. Coherent measures of risk. *Mathematical Finance*, 9(3):203–228, 1999.

[4] E. Bogentoft, H.E. Romeijn, and S. Uryasev. Asset/Liability management for pension funds using CVaR constraints. *Journal of Risk Finance*, 3(1):57–71, 2001.

[5] M. Broadie. Computing efficient frontiers using estimated parameters. *Annals of Operations Research*, 45:21–58, 1993.

[6] V. Chopra and W.T. Ziemba. The effects of errors in means, variances, and covariances on optimal portfolio choice. *The Journal of Portfolio Management*, Winter:6–11, 1993.

[7] K. Høyland, M. Kaut, and S.W. Wallace. A heuristic for moment matching scenario generation. *Computational Optimization and Applications*, 24(2-3):169–185, 2003.

[8] C.M. Jarque and A.K. Berra. Efficient tests for normality, homoscedasticity and derial independence of regression residuals. *Economic Letters*, 6(5):255–259, 1980.

[9] N. Jobst and S.A. Zenios. The tail that wags the dog: Integrating credit risk in asset portfolios. *Journal of Risk Finance*, 3(1):31–43, 2001.

[10] P. Jorion. *Value at Risk: The New Benchmark for Managing Financial Risk*. McGraw-Hill, New York, 2001.

[11] J.G. Kallberg and W.T. Ziemba. Mis-specifications in portfolio selection problems. In G. Bamberg and K. Spremann, editors, *Risk and Capital*, pages 74–87. Springer Verlag, 1984.

[12] M. Kaut and S.W. Wallace. Evaluation of scenario-generation methods for stochastic programming. *Pacific Journal of Optimization*, (forthcoming).

[13] P. Krokhmal, S. Uryasev, and G. Zrazhevsky. Risk management for hedge fund portfolios: A comparative analysis of linear portfolio rebalancing strategies. *Journal of Alternative Investments*, 5(1):10–29, 2002.

[14] P. Krokhmal, S. Uryasev, and G. Zrazhevsky. Numerical comparison of CVaR and CDaR approaches: Application to hedge funds. In W.T. Ziemba, editor, *The Stochastic Programming Approach to Asset-Liability and Wealth Management*. AIMR/Blackwell, 2003.

[15] R.T. Rockafellar and S. Uryasev. Conditional value-at-risk for general distributions. *Journal of Banking and Finance*, 26(7):1443–1471, 2002.

[16] D. Tasche. Expected shortfall and beyond. *Journal of Banking and Finance*, 26(7):1253–1272, 2002.

[17] N. Topaloglou, H. Vladimirou, and S.A. Zenios. CVaR models with selective hedging for international asset allocation. *Journal of Banking and Finance*, 26(7):1535–1561, 2002.

[18] N. Topaloglou, H. Vladimirou, and S.A. Zenios. A dynamic stochastic programming modeld for international portfolio management. *European Journal of Operational Research*, (in print).

Appendix: Properties of the data

Tables 3 and 4 present the first four marginal moments and the correlation matrix of the monthly differentials in the historical market data of the random variables (returns of the stock and bond indices, as well as of the spot currency exchange rates). These statistics constitute the targets matched in the scenario sets used in the empirical tests of section 4. Note that the random variables have skewness ranging from -1.00 to 1.36 and kurtosis ranging from 2.78 to 7.39. The historical observations indicate that the random variables in the international portfolio management problem are not normally distributed — Jarque-Berra tests [8] reject the normality hypothesis for these data (see, Topaloglou et al. [18]). This was a primary factor behind modelling choices in this study. That is,

1. We adopted the `CVaR` risk measure as it is suitable to accommodate higher-order moments and measures risk in the tail of the portfolio's return distribution.

2. We employ a scenario generation method based on principles of moment-matching as it provides full control of the moments in the generation of scenarios.

	Stk.USA	Stk.UK	Stk.Ger	Stk.Jap	Bnd1.USA	Bnd7.USA	Bnd1.UK
Mean	0.01296	0.01047	0.01057	-0.00189	0.00553	0.00702	0.00718
StDev	0.04101	0.04150	0.05796	0.06184	0.00467	0.01620	0.00688
Skew	-0.47903	-0.19051	-0.47281	0.04768	-0.18341	-0.07482	1.36036
Kurt	3.76519	3.11399	4.11970	3.62119	2.77801	3.23974	7.38764

Bnd7.UK	Bnd1.Ger	Bnd7.Ger	Bnd1.Jap	Bnd7.Jap	ExR.UK	ExR.Ger	ExR.Jap
0.00894	0.00535	0.00671	0.00318	0.00622	-0.00077	-0.00152	0.00179
0.01884	0.00455	0.01368	0.00506	0.01681	0.02801	0.03021	0.03607
0.12127	0.55214	-0.87820	0.54803	-0.53562	-0.99772	-0.25505	1.09286
3.52858	5.13927	4.42483	4.28775	5.23964	6.51592	3.80887	6.75996

Table 3: Moments of monthly differentials of the historical market data.

Tables 5 and 6 present the lengths of the *variation intervals* for the moments and correlations. The *variation intervals* are defined and used in the model sensitivity tests in Section 4.2.

	Stk.USA	Stk.UK	Stk.Ger	Stk.Jap	Bnd1.USA	Bnd7.USA	Bnd1.UK
Stk.UK	0.6651						
Stk.Ger	0.5573	0.5911					
Stk.Jap	0.3568	0.3601	0.3429				
Bnd1.USA	0.1965	0.0844	-0.0578	-0.0105			
Bnd7.USA	0.2656	0.1150	0.0027	0.0205	0.8768		
Bnd1.UK	0.0853	0.4014	0.0276	0.0018	0.3600	0.3176	
Bnd7.UK	0.2258	0.5075	0.1714	0.0392	0.4314	0.4815	0.8175
Bnd1.Ger	0.0556	0.2642	0.0536	0.0081	0.3466	0.3574	0.6121
Bnd7.Ger	0.1687	0.3066	0.2326	0.0408	0.4385	0.5453	0.4639
Bnd1.Jap	0.0557	0.0814	-0.0005	0.0226	0.2513	0.2186	0.3274
Bnd7.Jap	0.0463	0.0493	0.0140	-0.0029	0.2831	0.3235	0.1815
ExR.UK	0.0247	-0.2177	-0.1062	0.1162	0.2422	0.1911	-0.2811
ExR.Ger	-0.0643	-0.2263	-0.2651	-0.0828	0.2716	0.2129	-0.1429
ExR.Jap	0.1126	0.0945	-0.1414	0.0475	0.1319	0.0975	0.0927

	Bnd7.UK	Bnd1.Ger	Bnd7.Ger	Bnd1.Jap	Bnd7.Jap	ExR.UK	ExR.Ger
Bnd1.Ger	0.5688						
Bnd7.Ger	0.6627	0.7779					
Bnd1.Jap	0.2645	0.4008	0.2853				
Bnd7.Jap	0.2100	0.3025	0.3093	0.7827			
ExR.UK	-0.1588	-0.2227	-0.0948	0.0145	0.0262		
ExR.Ger	-0.1332	-0.0638	-0.0598	0.1021	0.1375	0.6949	
ExR.Jap	0.0680	0.0825	-0.0072	0.0334	-0.0122	0.2680	0.4236

Table 4: Correlations of monthly differentials of the historical market data.

	Stk.USA	Stk.UK	Stk.Ger	Stk.Jap	Bnd1.USA	Bnd7.USA	Bnd1.UK
Mean	0.00946	0.00764	0.01836	0.01055	0.00166	0.00435	0.00351
StDev	0.02035	0.01217	0.02153	0.02321	0.00171	0.00293	0.00485
Skew	1.62256	0.8916	1.0513	0.54337	0.79795	0.62041	2.32511
Kurt	3.87492	1.82066	2.56178	0.99143	1.2739	0.891	7.2035

Bnd7.UK	Bnd1.Ger	Bnd7.Ger	Bnd1.Jap	Bnd7.Jap	ExR.UK	ExR.Ger	ExR.Jap
0.00591	0.00366	0.00284	0.00433	0.00487	0.0057	0.0107	0.01108
0.00952	0.00174	0.00262	0.00249	0.00351	0.01533	0.00954	0.0132
0.78578	1.18387	0.55199	1.21049	1.48868	1.62235	1.22593	1.55678
1.04044	3.12805	2.33646	2.18066	5.13353	6.35287	2.086	4.0713

Table 5: Lengths of the *variation intervals* of moments.

	Stk.USA	Stk.UK	Stk.Ger	Stk.Jap	Bnd1.USA	Bnd7.USA	Bnd1.UK
Stk.UK	0.28245						
Stk.Ger	0.41613	0.1814					
Stk.Jap	0.33692	0.27907	0.38272				
Bnd1.USA	0.48044	0.43639	0.3613	0.21907			
Bnd7.USA	0.43832	0.44669	0.37637	0.28547	0.07029		
Bnd1.UK	0.25762	0.63827	0.47922	0.29948	0.32842	0.3272	
Bnd7.UK	0.18449	0.53845	0.3226	0.18788	0.30047	0.29125	0.10638
Bnd1.Ger	0.32837	0.53526	0.44861	0.20951	0.23945	0.21754	0.23587
Bnd7.Ger	0.28289	0.4632	0.52714	0.18048	0.25918	0.25712	0.1254
Bnd1.Jap	0.39658	0.30524	0.38022	0.47467	0.35529	0.34925	0.42718
Bnd7.Jap	0.53725	0.42657	0.43082	0.53627	0.1201	0.19085	0.34086
ExR.UK	0.32612	0.2294	0.23331	0.34925	0.37893	0.34466	0.48376
ExR.Ger	0.34274	0.23943	0.30395	0.41129	0.4128	0.42665	0.49304
ExR.Jap	0.35474	0.3389	0.22861	0.29166	0.23326	0.21832	0.19624

	Bnd7.UK	Bnd1.Ger	Bnd7.Ger	Bnd1.Jap	Bnd7.Jap	ExR.UK	ExR.Ger
Bnd1.Ger	0.18615						
Bnd7.Ger	0.19871	0.08346					
Bnd1.Jap	0.39379	0.41627	0.31011				
Bnd7.Jap	0.39649	0.35563	0.26651	0.11535			
ExR.UK	0.41516	0.57385	0.39195	0.26009	0.36195		
ExR.Ger	0.43402	0.33271	0.3127	0.22496	0.23136	0.34187	
ExR.Jap	0.18262	0.24788	0.25119	0.44653	0.5184	0.32304	0.27371

Table 6: Lengths of the *variation intervals* of correlations.